Problem Set : Linear models

Abhi's solutions
- Not Larry approved.

Input matrix is productive if for some $x^* \geq 0$,
$x^* \gg Ax^*$.

$\boxed{1}$ a) $A = \begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.3 \end{bmatrix}$

$$x^* = (1, 1, 1)^T \gg$$
$$Ax^* = (0.9, 0.9, 0.9)^T$$

So, A is productive.

b) $A = \begin{bmatrix} 0.6 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$

$$x \gg Ax = \begin{bmatrix} 0.6x_1 + 0.5x_2 \\ 0.1x_1 + 0.5x_2 \end{bmatrix}$$

$$\left.\begin{array}{l} x_1 > 0.6x_1 + 0.5x_2 \\ x_1 > \frac{5}{4}x_2 \end{array}\right\} (1)$$

$$\left.\begin{array}{l} x_2 > 0.1x_1 + 0.5x_2 \\ x_1 < 5x_2 \end{array}\right\} (2)$$

Productive

So $\frac{5}{4}x_2 < x_1 < 5x_2 \Rightarrow x^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $Ax^* = \begin{bmatrix} 1.7 \\ 0.7 \end{bmatrix}$.

$\boxed{2}$ TS: A is productive iff $(I-A)^{-1}$ exists and is non-negative.

Pf: " $\Longrightarrow$ "

Use Thm: A is productive $\Rightarrow$ $(I-A)$ has full rank.

Thus, $(I-A)^{-1}$ exists.

Use Thm: A is productive $\Rightarrow$ $\forall y \geq 0$,
$x = (I-A)^{-1} y \geq 0$

$e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \geq 0 \Rightarrow x = (I-A)^{-1} e_i \geq 0 \quad \forall i$

Thus, each column of A is non-negative.

" $\Longleftarrow$ "

If $(I-A)^{-1}$ exists and is non-negative,

$\vec{u} = (1, \ldots, 1) \gg 0$

$\bar{x} = (I-A)^{-1} \vec{u} \geq 0.$

$(I-A)\bar{x} = u \gg 0$

So, $\exists \ \vec{x} \geq 0$ where $\bar{x} \gg A\bar{x}$

$\Rightarrow$ A is productive.

3. TS: $A_{n\times n}$ is productive $\Rightarrow \exists\, 1 \leq j \leq n : (A\cdot \iota)_j < 1$

$$\iota = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Pf: Towards a contradiction:

Suppose: $A\iota \geq \iota$

$$(I-A)\iota = y \leq 0.$$

$$\exists\, x^* \geq 0 : (I-A)x^* \geq 0. \checkmark$$

$$\Rightarrow \exists\, \lambda \in [0,1] : \quad z = \lambda \iota + (1-\lambda)x^*$$

such that: $(I-A)z = 0.$, $z \neq 0$

$(\iota \geq 0, x^* \geq 0)$

$\Rightarrow (I-A)$ not full rank!

4. $\pi = p(I-A) - a_0$

In eqbm: $\pi = 0$

$$p^* = a^0 (I-A)^{-1}$$

$$(I-A)^{-1} = \begin{bmatrix} 5/2 & 9/2 & 7/2 \\ 5/3 & 19/3 & 7/3 \\ 5/6 & 16/6 & 7/6 \end{bmatrix}$$

$$a_0 (I-A)^{-1} = [5,\ 27/2,\ 9].$$

$\boxed{5}$ (a) Review section 4 notes and recordings.

(b) Notice that the proof in the notes sets

$$\vec{p} = a_0 [I-A]^{-1} \vec{y}$$

$\vec{p} \gg \vec{0}$ if $a_0 [I-A]^{-1} \gg 0$ since $\vec{y} \gg \vec{0}$.

We will show that $(I-A)^{-1} \gg 0$.

For a productive, A, we use a von-Neumann expansion:

$$(I-A)^{-1} = I + A + A^2 + \cdots$$

From (a) we know that $\forall i,j, \exists m$:

$$A^m_{ij} > 0$$

$$\Rightarrow \underline{\underline{(I-A)^{-1} \gg 0}}$$

$\boxed{6}$ 2 goods $\longrightarrow$ wine (v), mutton (m).

2 input requirement $\longrightarrow$ labor in Spain, labor in England.

(a) In such a model, the PPS should be

$$Y = \{\vec{y} \geq 0 : \vec{y} \leq (B-A)\vec{x}, \; A_0\vec{x} \leq \vec{L}, \; \vec{x} \geq 0\}$$

Here:

$$\vec{y} = (v, m)^T$$

$$\vec{x} = (x_{ve}, x_{vs}, x_{me}, x_{ms})^T$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is because producing v or m in any country only requires a labor input and no good input.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

So, $\vec{y} \leq (B-A)\vec{x} \Rightarrow \begin{array}{l} v \leq x_{ve} + x_{vs} \\ m \leq x_{me} + x_{ms} \end{array}$

$$A_0 = \begin{bmatrix} a_{ve} & 0 & a_{me} & 0 \\ 0 & a_{vs} & 0 & a_{ms} \end{bmatrix} \; ; \; \vec{L} = \begin{bmatrix} l_e \\ l_s \end{bmatrix}$$

(b) Convex polyhedron for PPS:

Y = the set of all (v, m) that solve:

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & a_{ve} & a_{me} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{vs} & a_{ms} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} v \\ m \\ x_{ve} \\ x_{me} \\ x_{vs} \\ x_{ms} \end{bmatrix}}_{\vec{z}} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ l_e \\ l_s \end{bmatrix}}_{\vec{r}}$$

where $v, m, x_{ve}, x_{me}, x_{vs}, x_{ms} \geq 0$.

Checking convexity of set:

suppose $v, m, x_{ve}, x_{me}, x_{vs}, x_{ms} \geq 0$ &

$v', m', x'_{ve}, x'_{me}, x'_{vs}, x'_{ms} \geq 0$

satisfy above inequalities, then:

$\forall \; \tilde{z} = \lambda z + (1-\lambda) z'$, $\tilde{z} \geq 0$

$$Q \cdot \tilde{z} = Q\left(\lambda \vec{z} + (1-\lambda)\vec{z}'\right)$$

$$= \lambda Q \vec{z} + (1-\lambda) Q \vec{z}' \leq \lambda \vec{r} + (1-\lambda)\vec{r}$$

$$\leq \vec{r} \;//$$

Checking closedness:

suppose $\exists \{\vec{z}_i : i \geqslant 1\}$ sequence such that

i - $\forall i$, $Q \cdot \vec{z}_i \leq \vec{r}$

ii - $\vec{z}_i \longrightarrow \vec{z}^*$

Since weak inequalities are preserved in the limit:

$$Q \cdot \vec{z}^* \leq \vec{r} \quad \text{and} \quad \vec{z}^* \geqslant \vec{0}$$

(c) $$h(\alpha, \beta) = \max_{\vec{z}} \ \alpha z_1 + \beta z_2$$

$$\text{s.t} \quad Q \cdot \vec{z} \leq \vec{r}$$
$$\vec{z} \geqslant 0$$

(d) Dual: $$\min_{p_v, p_m, w_e, w_s} \ w_e L_e + w_s L_s$$

s.t.

$$p_v \geqslant \alpha \qquad ---- (v)$$
$$p_m \geqslant \beta \qquad --- (m)$$

$$-p_v + w_e a_{ve} \geqslant 0 \qquad ---- (x_{ve})$$
$$-p_m + w_e a_{me} \geqslant 0 \qquad --- (x_{me})$$
$$-p_v + w_s a_{vs} \geqslant 0 \qquad --- (x_{vs})$$
$$-p_m + w_s a_{ms} \geqslant 0 \qquad --- (x_{ms})$$
$$p_v, p_m, w_e, w_s \geqslant 0$$

(e) From constraints 1 and 2:

$$v(p_v - \alpha) = m(p_m - \beta) = 0$$

If $(v, m) \gg 0 \Rightarrow \alpha = p_v$ and $\beta = p_m$.

If $i \in \{v, m\}$ and $j \in \{e, s\}$,

$$x_{ij} \cdot [w_j a_{ij} - p_i] = 0$$

If $w_j a_{ij} > p_i \Rightarrow$ negative profits

then $x_{ij} = 0$.

(f) $x_{ve} > 0$, $x_{vs} > 0$, $x_{me} > 0$, $x_{ms} > 0$

By complementary slackness:

i- $w_e a_{ve} = p_v = w_s a_{vs} \Rightarrow \dfrac{w_e}{w_s} = \dfrac{a_{vs}}{a_{ve}}$

ii- $w_e a_{me} = p_m = w_s a_{ms} \Rightarrow \dfrac{w_e}{w_s} = \dfrac{a_{ms}}{a_{me}}$

$$\Rightarrow \frac{a_{vs}}{a_{ms}} = \frac{a_{ve}}{a_{me}}$$

No comparative advantage.

(g) Similar to f, $\dfrac{a_{me}}{a_{ve}} \leqslant \dfrac{a_{ms}}{a_{vs}}$

$\boxed{7}$ Countries : A , B

Goods : $z \in [0,1]$

Labor reqt : $\big(a(z) \,,\, b(z)\big)_{z\in[0,1]}$ – only 1 input

z is ordered so that $\alpha(z) = a(z)/b(z)$ strictly $\downarrow$ in z.

Output prices : $\big(p(z)\big)_{z\in[0,1]}$

Input prices : w_a , w_b

Input endowments : L_A , L_B

$\boxed{a}$ Convex support function of world PPS :

$$v(L_A, L_B) = \max_{x_a(\cdot),\, x_b(\cdot)} \int_0^1 p(z)\cdot\left[x_a(z) + x_b(z)\right] dz$$

$$\text{s.t.} \quad \int_0^1 a(z)\, x_a(z)\, dz \leq L_A \quad \text{----}(w_a)$$

$$\int_0^1 b(z)\, x_b(z)\, dz \leq L_B \quad \text{----}(w_b)$$

$$x_a(\cdot),\, x_b(\cdot) \geqslant 0$$

Complementary slackness :

$$w_a\left[\int_0^1 a(z)\, x_a(z)\, dz - L_A\right] = 0$$

$$w_b\left[\int_0^1 b(z)\, x_b(z)\, dz - L_B\right] = 0$$

$$\text{Dual} = \min_{\omega_a, \omega_b} \omega_a L_A + \omega_b L_B$$

$$\text{s.t.} \left\{ \omega_a \cdot a(z) \geq p(z) \quad \ldots\ldots (x_a(z)) \right\}_{\forall z}$$

$$\left\{ \omega_b \cdot b(z) \geq p(z) \quad \ldots\ldots (x_b(z)) \right\}_{\forall z}$$

$$\omega_a, \omega_b \geq 0$$

Profit maximization condition: $\forall z \in [0,1]$,

$$x_a(z)\left[p(z) - \omega_a a(z)\right] = 0, \quad p(z) \leq \omega_a a(z)$$

$$x_b(z)\left[p(z) - \omega_b b(z)\right] = 0, \quad p(z) \leq \omega_b b(z)$$

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$\boxed{b}$ If $x_a(\tilde{z}) > 0$, $x_b(\tilde{z}) > 0$ for some $\tilde{z} \in [0,1]$

then:

$$p(\tilde{z}) = \omega_a a(\tilde{z})$$

$$p(\tilde{z}) = \omega_b b(\tilde{z})$$

$$\Rightarrow \frac{a(\tilde{z})}{b(\tilde{z})} = \alpha(\tilde{z}) = \underline{\underline{\frac{\omega_b}{\omega_a}}}$$

$\boxed{c}$ <u>Altered</u> proposition: $\exists z^* \in [0,1]$ such that any $x_A^*(z) > 0$ has $z \geq z^*$ & any $x_B^*(z) > 0$ has $z \leq z^*$.

Pf: Key ingredient:

i - Complementary slackness conditions

ii - $\alpha(\cdot)$ is strictly decreasing

Part (b) above says that for any good produced in both countries, $\alpha(\tilde{z}) = \frac{w_b}{w_a}$.

Since, $\alpha(\cdot)$ is strictly decreasing

$\Rightarrow \exists$ atmost one $\tilde{z} \in [0,1] : \alpha(\tilde{z}) = \frac{w_b}{w_a}$.

Towards a contradiction, suppose

$\exists\, z_1, z_2$ such that $z_1 > z_2$.

Let A produces z_2 and B produces z_1.

$$w_a a(z_1) \geq p(z_1) \quad \& \quad w_b b(z_1) = p(z_1)$$

$$w_a a(z_2) = p(z_2) \quad\quad w_b b(z_2) \geq p(z_2)$$

$$\frac{a(z_1)}{a(z_2)} \geq \frac{p(z_1)}{p(z_2)} \quad \& \quad \frac{p(z_1)}{p(z_2)} \geq \frac{b(z_1)}{b(z_2)}.$$

$$\Rightarrow \frac{a(z_1)}{b(z_1)} \geq \frac{a(z_2)}{b(z_2)} \equiv \alpha(z_1) \geq \alpha(z_2)$$

By strict decreasing: $\alpha(z_1) < \alpha(z_2) \Rightarrow$ Contradiction

But suppose $x_a(z_1) > 0$, does this imply $x_a(z_2) > 0 \;\; \forall z_2 > z_1$?

$x_a(z_1) > 0 \Rightarrow p(z_1) = w_a a(z_1)$.

It is possible that:

$$p(z_2) < w_a a(z_2) \Rightarrow x_a(z_2) = 0$$

$$\frac{p(z_2)}{a(z_2)} < \frac{p(z_1)}{a(z_1)}. \;!$$

$\boxed{d}$ See $\underline{b}$. Notice that z^* is the only good that both A and B can produce simultaneously. This happens when: $0 \leq z^* = \frac{w_b}{w_a} \leq 1$

$\boxed{e}$ Identical consumers with Cobb-Douglas preferences

$\Rightarrow$ Use aggregate representation.

Consumer problem: $$\max_{C_A} \int_0^1 U_A(C_A(z))\,dz$$

$$\text{s.t. } \int_0^1 p(z) C_A(z)\,dz \leq w_A L_A$$

$U_A(\cdot)$ is Cobb Douglas

$$\Rightarrow \text{LNS} \Rightarrow \int_0^1 p(z)\, c_A(z)\, dz = w_A L_A.$$

$$\therefore \int_0^1 \underbrace{p(z) \frac{c_A(z)}{w_A L_A}}_{\equiv s_A(z)} dz = 1.$$

If $U_A = U_B$, given preferences are C-D:

$$\frac{p(z)\, c_A(z)}{w_A L_A} = \frac{p(z)\, c_B(z)}{w_B L_A}$$

$$s_A(z) = s_B(z) = s(z).$$

$\boxed{f}$ Fraction spent on country A goods:

$$\tilde{\theta}(z^*) = \int_{z^*}^1 s(z)\, dz \qquad \left(\because \tilde{\theta} = \underset{\text{in p.s.}}{1-\theta}\right)$$

Revenue in A $= \tilde{\theta}(z^*)\,[w_a L_a + w_b L_b]$

Labor expense in A $= w_a L_a$

Eqbm: $\tilde{\theta}(z^*)\, w_b L_b = w_a L_a\,[1 - \tilde{\theta}(z^*)]$

Revenue in B $= \left(1-\tilde{\theta}(z^*)\right)\left[w_a L_a + w_b L_b\right]$

Labor expense in B $= w_b L_b$

Eqbm: $\tilde{\theta}(z^*)\, w_b L_b = \left(1-\tilde{\theta}(z^*)\right) w_a L_a.$

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$\boxed{g}$

$$\frac{w_b}{w_a} = \frac{L_a}{L_b}\,\frac{\left(1-\tilde{\theta}(z^*)\right)}{\tilde{\theta}(z^*)} \equiv \beta(z^*)$$

$\boxed{h}$

$$\beta(z^*) = \frac{w_b}{w_a} = \alpha(z^*).$$

Diff. w.r.t. L_a

$$\alpha'(z^*)\cdot\frac{dz^*}{dL_a} = \frac{d}{dL_a}\left\{\frac{L_a}{L_b}\,\frac{\left(1-\tilde{\theta}(z^*)\right)}{\tilde{\theta}(z^*)}\right\}$$

$$= \frac{1}{L_b}\left[\frac{1-\tilde{\theta}(z^*)}{\tilde{\theta}(z^*)}\right] + \frac{L_a}{L_b}\left[\frac{\left(-\tilde{\theta}(z^*)-(1-\tilde{\theta}(z^*))\right)\frac{d\tilde{\theta}(z^*)}{dL_a}}{\tilde{\theta}(z^*)^2}\right]$$

$$= \frac{1}{L_b}\left[\frac{1-\tilde{\theta}(z^*)}{\tilde{\theta}(z^*)}\right] - \frac{L_a}{L_b}\left(\frac{d\tilde{\theta}(z^*)/dz^*\cdot dz^*/dL_a}{\tilde{\theta}(z^*)^2}\right)$$

$$\tilde{\theta}(z^*) = \int_{z^*}^{1} s(z)\,dz$$

Using Leibniz Rule:

$$\frac{d\tilde{\theta}(z^*)}{dz^*} = -1 \cdot s(z^*)$$

$$\therefore \alpha'(z^*)\frac{dz^*}{dL_a} = \frac{1}{L_b}\left[\frac{1-\tilde{\theta}(z^*)}{\tilde{\theta}(z^*)}\right] + \frac{L_a}{L_b}\left[\frac{s(z^*)}{\tilde{\theta}(z^*)^2}\right]\frac{dz^*}{dL_a}$$

$$\alpha'(z^*) L_b \tilde{\theta}(z^*)^2 \frac{dz^*}{dL_a} = \tilde{\theta}(z^*)(1-\tilde{\theta}(z^*)) + L_a\, s(z^*)\frac{dz^*}{dL_a}$$

$$\frac{dz^*}{dL_a} = \frac{\tilde{\theta}(z^*)\left[1-\tilde{\theta}(z^*)\right]}{\underbrace{\alpha'(z^*)}_{<0} L_b \tilde{\theta}(z^*)^2 - L_a \underbrace{s(z^*)}_{>0}}$$

$$\frac{dz^*}{dL_a} < 0$$

Intuition: $L_a \uparrow \Rightarrow w_a \downarrow \Rightarrow \frac{w_b}{w_a}\uparrow$,

$\alpha(z^*) = \frac{w_b}{w_a}$ & $\alpha'(\cdot) < 0$

$\Rightarrow z^* \downarrow$

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