Problem Set 3: Two-sector model

[1] Cost minimization:

$$c(w, r, q) = \min_{l,k} \quad w \cdot l + r \cdot k$$

$$\text{s.t.} \quad f(l,k) \geq q.$$
$$l, k \geq 0.$$

a) $f(k, l) = A \cdot k^{\alpha} l^{\beta}$

$$\mathcal{L}(k, l) = w \cdot l + r \cdot k + \lambda\left(A \cdot k^{\alpha} l^{\beta} - q\right)$$

$$\text{FOC}(k) \Rightarrow r = \lambda \alpha A l^{\beta} k^{\alpha-1} \quad \text{---} (1)$$

$$\text{FOC}(l) \Rightarrow w = \lambda \beta A k^{\alpha} l^{\beta-1} \quad \text{---} (2)$$

$$\text{Constraint} \Rightarrow q = A k^{\alpha} l^{\beta}$$

From (1) & (2):

$$\frac{r}{w} = \frac{\lambda \alpha\, q/k}{\lambda \beta\, q/l} = \frac{\alpha l}{\beta k}$$

$$\therefore \quad l^{*} = \frac{\beta}{\alpha} \cdot \frac{r}{w} \cdot k^{*}$$

Plug into constraint:

$$A k^{\alpha}\left[\frac{\beta}{\alpha}\cdot\frac{r}{w}\cdot k\right]^{\beta} = q$$

$$k^{\alpha+\beta}\,\alpha^{-\beta}\,\beta^{\beta}\left(\frac{r}{w}\right)^{\beta}\cdot A = q$$

$$\therefore\ k^{*} = \left(\frac{\alpha^{\beta} q\, w^{\beta}}{A\,\beta^{\beta} r^{\beta}}\right)^{1/(\alpha+\beta)}$$

$$l^{*} = \frac{\beta}{\alpha}\,\frac{r}{w}\left[\frac{q\,(\alpha w)^{\beta}}{A\alpha'\,(\beta r)^{\beta}}\right]^{\frac{1}{\alpha+\beta}}$$

$$= \left(\frac{q\,(\beta r)^{\alpha}}{A\,(\alpha w)^{\alpha}}\right)^{\frac{1}{\alpha+\beta}}$$

$$c(w, r, q) = w\cdot l^{*} + r\cdot k^{*}$$

$$= \frac{w^{\beta} q\,(\beta r)^{\alpha}}{A\,\alpha^{\alpha}} + \frac{r^{\alpha} q\,(\alpha w)^{\beta}}{A\,\beta^{\beta}}$$

I won't do the derivations of all in the interest of brevity.

(b) $c(w, r, q) = q\left[\left(\frac{1-a}{w^{\rho}}\right)^{\frac{1}{1-\rho}} + \left(\frac{a}{r^{\rho}}\right)^{\frac{1}{1-\rho}}\right]^{\frac{\rho-1}{\rho}}$

(c) $c(w, r, q) = q \min\{w/b, r/a\}$

(d) $c(w, r, q) = q\left(\frac{r}{a} + \frac{w}{b}\right)$

(e) $c(w, r, q)$

$$= \mathbb{1}\{0<q<1\}\left[-\left(\frac{r}{a}\log\left(1-x^*\right) + \frac{w}{b}\log\left(1-y^*\right)\right)\right]$$

where $x^* \cdot y^* = q$ and

$$x^* = \frac{q(rb - wa) + \sqrt{q^2(wa-rb)^2 + 4warbq}}{2rb}$$

$\boxed{2}$ Prices : P_A , P_B

Technologies : f_A , f_B

(a) To show: in any equilibrium,

$$wK + rL = P_A Y_A + P_B Y_B .$$

Pf: Any equilibrium in a HOV model satisfies the following condition:

$$y_A \left(P_A - c_A(r, w)\right)) = 0$$

$$y_B \left(P_B - c_B(r, w)\right) = 0$$

$$K = y_A \cdot \frac{\partial c_A(r,w)}{\partial r} + y_B \frac{\partial c_B(r,w)}{\partial r}$$

$$L = y_A \cdot \frac{\partial c_B(r,w)}{\partial w} + y_B \frac{\partial c_B(r,w)}{\partial w}$$

$$\text{Revenue} = P_A Y_A + P_B Y_B$$

$$= c_A(r,w)\, y_A + c_B(r,w)\, y_B$$

$$= C_A(r, w, y_A) + C_B(r, w, y_B)$$

$$= \underline{wL + rK} .$$

(b) Capital share of national product:

$$\theta = \frac{r.K}{P_A Y_A + P_B Y_B}$$

From (a): $P_A Y_A + P_B Y_B = rK + wL$

$$\theta = \frac{r.K}{rK + wL}$$

$$= \frac{K}{K + L\left(\frac{w}{r}\right)}$$

Assume (L, K) is within the cone of diversity, then, the Stopler-Samuelson theorem gives us that:

i - If A is capital intensive:

$$P_A \uparrow \Rightarrow \frac{w}{r} \uparrow \Rightarrow \theta \downarrow$$

ii - If A is labor intensive:

$$P_A \uparrow \Rightarrow \frac{w}{r} \downarrow \Rightarrow \theta \uparrow.$$

3 Think of the equilibrium conditions:

$$y_A \frac{\partial c_A(r,w)}{\partial r} + y_B \frac{\partial c_B(r,w)}{\partial r} = K$$

$$y_A \frac{\partial c_A(r,w)}{\partial w} + y_B \frac{\partial c_B(r,w)}{\partial w} = L$$

In short, $(K,L) = y_A \nabla c_A + y_B \nabla c_B$

Assume $K' = K + \varepsilon$ for small $\varepsilon > 0$

$$(K', L) = \tilde{y}_A \nabla c_A + \tilde{y}_B \nabla c_B.$$

Using IFT:

$$\overline{y}_A = \frac{\begin{vmatrix} K' & \partial c_B/\partial r \\ L & \partial c_B/\partial w \end{vmatrix}}{|D|}$$

where $|D| = \frac{\partial c_A}{\partial r}\frac{\partial c_B}{\partial w} - \frac{\partial c_A}{\partial w}\frac{\partial c_B}{\partial r}$

$$= y_A + \frac{\partial c_B}{\partial w}\varepsilon \Big/ |D|.$$

$$\overline{y}_B = \begin{vmatrix} \partial c_A/\partial r & K' \\ \partial c_A/\partial w & L \end{vmatrix} \Big/ |D|$$

$$= y_B - \frac{\partial c_A}{\partial w}\varepsilon \Big/ |D|.$$

Now, $|D| > 0$ iff $\frac{\partial c_A/\partial r}{\partial c_A/\partial w} > \frac{\partial c_B/\partial r}{\partial c_B/\partial w}$

$$\iff \frac{k_A}{l_A} > \frac{k_B}{l_B}$$

$$\Rightarrow \tilde{y}_A > y_A \ \& \ \tilde{y}_B < y_B$$

$$|D| < 0 \iff \frac{k_A}{l_A} < \frac{k_B}{l_B}$$

$$\Rightarrow \tilde{y}_A < y_A \ \& \ \tilde{y}_B > y_B .$$

$\boxed{4}$ (a) $y_i = Y_i/L_i = F_i(K_i, L_i)/L_i$

(HOD 1) $= F_i(K_i/L_i, 1)$

$= f_i(k_i)$

$f_i'(k_i) > 0$ and $f_i''(k_i) < 0$

$\Rightarrow f_i(\cdot)$ is strictly increasing, concave & C^2.

$\lim_{K_i \to 0} f_i(K_i/L_i) = \infty$, Inada condition holds.

$f_i(\alpha k_i) = F_i(\alpha K_i/L_i, 1) \neq \alpha f(k_i)$

$\Rightarrow$ Not HOD 1.

(b) $P_i\, \partial F_i/\partial K_i = r$ and $P_i\, \partial F_i/\partial L_i = w$
says each factor is paid its marginal product.

$K_1 + K_2 = K$ and $L_1 + L_2 = L$
says the constraint for each factor binds.

$P_1 Y_1 = rK$ and $P_2 Y_2 = wL$
are the zero-profit conditions.

(c) + $Y_i = F_i(K_i, L_i) \longrightarrow y_i = f_i(k_i)$

since $\dfrac{Y_i}{L_i} = \dfrac{F_i(K_i, L_i)}{L_i} \underset{\text{HOD1}}{=} F_i(K_i/L_i, 1)$

(Also note: $\partial Y_i/\partial K_i = \partial F_i(K_i, L_i)/\partial K_1 = \partial F_i(K_i/L_i, 1)/\partial K_i = f_i'(k_i)$)

$$+ \quad \omega = \frac{w}{r} = \frac{P_i \, \partial F_i / \partial L_i}{P_i \, \partial F_i / \partial K_i}$$

$$\omega = \frac{\partial F_i(K_i, L_i) / \partial L_i}{\partial F_i(K_i, L_i) / \partial K_i}$$

By HOD1: $\quad K_i \, \partial F_i/\partial K_i + L_i \, \partial F_i / \partial L_i = F_i$

$$\therefore \omega = \frac{F_i/L_i - K_i/L_i \, \partial F_i/\partial K_i}{\partial F_i/\partial K_i}$$

$$= \frac{f_i(k_i) - k_i \cdot f'(k_i)}{f_i'(k_i)}$$

$$= \frac{f_i(k_i)}{f_i'(k_i)} - k_i$$

$$+ \quad K_1 + K_2 = K \longrightarrow \frac{K_1 + K_2}{L} = k$$

$$\Rightarrow \frac{K_1 L_1}{L_1 L} + \frac{K_2 L_2}{L_2 L} = k \Rightarrow k_1 l_1 + k_2 l_2 = k$$

$$L_1 + L_2 = L \longrightarrow l_1 + l_2 = 1$$

$+ \; P_1 Y_1 = rK \;$ & $\; r = P_1 \, \partial F_1 / \partial K_1$

$$\Rightarrow P_1 = \frac{r}{\partial F_1 / \partial K_1}$$

$$\therefore \quad \frac{Y_1 \cdot \cancel{q}}{\partial F_1/\partial K_1} = \cancel{q} K$$

$$F_1(K_1, L_1) = K \cdot \partial F_1/\partial K_1$$

Divide by $L_1 + L_2$ and use $F_1(K_1, L_1) = L_1 F_1(K_1/L_1, 1)$

$$l_1 f_1(k_1) = k \cdot f_1'(k_1)$$

Similarly, $P_2 Y_2 = \omega L$ and $w = P_2 \, \partial F_2/\partial L_2$

$$F_2(K_2, L_2) = Y_2 = L \; \partial F_2/\partial L_2$$

$$= L \left\{ \frac{1}{L_2} F_2(K_2, L_2) - \frac{K_2}{L_2} \frac{\partial F_2}{\partial K_2} \right\}$$

$$L_2 \; f_2(k_2) = L \left\{ f_2(k_2) - k_2 f_2'(k_2) \right\}$$

$$l_1 f_2(k_2) = k_2 f_2'(k_2) \qquad \left(\because 1 - \frac{L_2}{L} = l_1 \right)$$

(d) $$\omega = \frac{f_i(k_i)}{f_i'(k_i)} - k_i$$

$$d\omega = \frac{f_i'(k_i)}{f_i'(k_i)}\, dk_i - \frac{f_i(k_i) \cdot f_i''(k_i)\, dk_i}{\left(f_i'(k_i)\right)^2} - dk_i$$

$$\frac{dk_i}{d\omega} = -\frac{\left[f_i'(k_i)\right]^2}{f_i(k_i)\, f_i''(k_i)} > 0$$

k_i is strictly increasing in $\omega \Rightarrow \omega^{-1}(k_i)$ is uniquely determined.

(e) From (c): setting $l_1 = l_1$ in the 2 formulae:

$$\frac{k\, f_1'(k_1)}{f_1(k_1)} = \frac{k_1\, f_2'(k_2)}{f_2(k_2)}$$

$$\omega = \frac{w}{r} = \frac{f_i(k_i)}{f_i'(k_i)} - k_i$$

$$\frac{k}{\omega + k_1} = \frac{k_1}{\omega + k_2}$$

$$k = \frac{\omega + k_1}{\omega + k_2} \cdot k_2$$