ECON 6100 4/30/2021

Section 10

Lecturer: Larry Blume *TA: Abhi Ananth*

* Adopted from Fikri Pitsuwan's notes.

1 Review

The Transferable Utility (TU) Matching Problem consists of a set of L workers and F firms. We also denote the set of workers and firms by L and F, respectively. For each $l \in L$ and $f \in F$,

Feasability: $\forall l, \sum_f x_{lf} \leq 1$, $\forall f, \sum_l x_{lf} \leq 1$

- v_{lf} = surplus that l and f can generate when they are matched (given).
- $x_{lf} = 1$ if l and f matched, 0 otherwise (the TU matching problem is to find this). Note that each l and only be matched with one f and vice versa. We call x the match vector.
- If l is matched with f then they split the surplus $v_{lf} = w_l + \pi_f$.
- We call $(x, w, \pi) \in \{0,1\}^{LF} \times \mathbb{R}^L_+ \times \mathbb{R}^F_+$ the matching allocation.

Example 1. We have 2 firms and 2 workers with surplus given in the table, e.g., $v_{XA} = 4$ and so on. The match vector is $x = (x_{XA}, x_{XB}, x_{YA}, x_{YB}) \in \{0,1\}^4$. There are two possible match vectors $x' = (1,0,0,1)$ and $x'' = (0,1,1,0)$.

Workers \ Firms	A	B
X	4	5
Y	3	6

Figure 1: Example 1

	A	B
X	2, 2	1, 4
Y	1, 2	3, 3

X-A, Y-B does P.D.

Y-A, X-B.

1.9 4.1

With TU: PO $\Longleftrightarrow$ maximizing sum of value

What properties should a matching allocation satisfy? The most natural property would be that it maximizes the total surplus generated among $L \cup F$. We shall refer to this as *optimality*.

Definition 1. An allocation is *optimal* if it maximizes total surplus generated among $L \cup F$.

Example 2. Allocation with $x' = (1,0,0,1)$ is optimal since it yields a surplus of $4 + 6 = 10$, while $x'' = (0,1,1,0)$ yields $5 + 3 = 8$.

w_l, π_f $l \leftrightarrow f$ it generates $v_{lf} = w_l + \pi_f$.

The next natural question is how should we split the surplus among workers and firms? The answer to this involves a notion we refer to as *stability*.

Definition 2. An allocation is *stable* if no worker and firm that are not matched together can increase their welfare by matching with each other and dividing the surplus among themselves.

Example 3. Consider $x' = (1,0,0,1)$ and $w_X = 1$, $\pi_A = 3$, $w_Y = 5$, $\pi_B = 1$. Then X and B would be better off matching together. If the surplus are $w_X = 2$, $\pi_A = 2$, $w_Y = 2$, $\pi_B = 4$ then X and B can not both do better by matching together, so do Y and A. This is stable.

CE: w^*, π^* $l \leftrightarrow f$. (i) $\forall f' \neq f$: $w^*_l + \pi^*_{f'} \geq v_{lf'}$

For f : $w^*_l + \pi^*_f = v_{lf}$.

It turns out that optimality and stability intertwine in a strong sense in TU matching. To see this connection, we turn to linear programming. To find the optimal match we solve the following LP:

$$v(L \cup F) = \max_x \sum_{l,f} v_{lf} x_{lf} \qquad \text{s.t.} \quad \text{for all } l \;\; \sum_f x_{lf} \leq 1 \quad \text{for all } f \;\; \sum_l x_{lf} \leq 1 \quad \text{for all } l,f \;\; x_{lf} \geq 0$$

The solution to the primal tells you how to match workers and firms. Now, how should the surplus be divided to ensure stability? We look at the dual problem:

$$\min_{w,\pi} \sum_l w_l + \sum_f \pi_f \qquad \text{s.t.} \quad \text{for all } l,f \quad w_l + \pi_f \geq v_{lf} \qquad w,\pi \geq 0$$

We claim that the constraints of the dual ensure stability. Suppose X is matched with A and they split $w_X + \pi_A = v_{XA}$. If $w_X + \pi_C \geq v_{XC}$ for all C, then if X is matched with C', they are splitting $v_{XC'}$ among them, so X and C' cannot both gain from deviating from their current match. This leads us to a more specific definition of stability in TU matching.

Definition 3. We say that (x, w, π) is a *stable allocation* if (i) for all $l \in L$ and $f \in F$, $w_l + \pi_f \geq v_{lf}$, and (2) if $x_{lf} = 1$ then $w_l + \pi_f = v_{lf}$.

Example 4. $x' = (1, 0, 0, 1)$ solves the primal problem, now knowing this and the constraints of the dual we have $w_X + \pi_A = 4$, $w_X + \pi_B \geq 5$, $w_Y + \pi_A \geq 3$, and $w_Y + \pi_B = 6$. The values in the previous example give a stable allocation (or fail to do so) because they satisfy (or violate) these conditions.

LP Duality theorem leads to the following theorem.

Theorem 1. *x is optimal if and only if there are (w, π) such that (x, w, π) is a stable allocation.*

If we impose some structure on the surplus that each match can generate, can we say more about the optimal and stable allocation? Think of the surplus of each match as generated by a function $v : L \times F \to \mathbb{R}$. That is, $v_{lf} = v(l, f)$. Further if we suppose that workers in L and firms in F can be ranked, what condition on $v(\cdot, \cdot)$ guarantees that the match is (positive) *assortative*, that is highly ranked workers get matched with highly ranked firms?

$1 > 2$ $\quad$ $l'f'$ $\quad$ l, f'

Theorem 2. *If function v has increasing differences, that is for all $x' > x$ and $y' > y$ implies $v(x', y') - v(x, y') \geq v(x', y) - v(x, y)$, then every stable match is assortative.*

l', f $\quad$ l, f $\qquad\qquad$ $l' \succ_F l \quad f' \succ_L f$

2 Problems

Problem 1. Suppose we have 2 workers X, Y and 2 firms A, B. The workers and firms can be ranked unambiguously so that X is more productive than Y and A is more productive than B. More precisely, we have $v_{XA} > v_{XB}, v_{YA} > v_{YB}, v_{XA} > v_{YA}$, and $v_{XB} > v_{YB}$.

(a) Show that if positive assortative matching is stable, then $\boxed{v_{XA} - v_{YA} \geq v_{XB} - v_{YB}}$.

(b) Give a similar condition for when negative assortative matching is stable. → $v_{XA} - v_{YA} \leq v_{XB} - v_{YB}$

(c) If the condition of part (a) is satisfied striclty, can the negative assortative matching be stable?

(d) Find the values $v_{XA}, v_{XB}, v_{YA}, v_{YB}$ so that both positive assortative matching and negative assortative matching are stable.

(a) Positive assort. match: $X-A$, $Y-B$ $\qquad v, w, \pi$

$x_{XA} = 1$, $x_{YB} = 1$

$$x_{ef}(w_e + \pi_f - v_{ef}) = 0 \ ; \quad w_e + \pi_f \geq v_{ef}.$$

$$w_X + \pi_A = v_{XA} \ ; \quad w_X + \pi_B \geq v_{XB}$$

$$\boxed{w_Y + \pi_A} \geq v_{YA} \ ; \quad w_Y + \pi_B = v_{YB}$$

$$\begin{aligned} v_{XA} - v_{YA} &= w_X + \pi_A - v_{YA}. \\ &\geq w_X + \cancel{\pi_A} - w_Y - \cancel{\pi_A} \pm \pi_B \\ &= \underbrace{w_X + \pi_B}_{\geq v_{XB}} - \underbrace{(w_Y + \pi_B)}_{v_{YB}}. \\ &\geq v_{XB} - v_{YB}. \end{aligned}$$

(b) Neg. Assort. Match: $X - B$, $Y - A$

$x_{XB} = 1$, $x_{YA} = 1$ Corr = - 1

$v_{XA} - v_{YA} \le v_{XB} - v_{YB}$. $\longleftrightarrow$

Proof: is identical.

(d)

V	A	B
X	1	1
Y	0	0

$v_{YA} = v_{YB} = 0$

$v_{XA} = v_{XB} = 1$.

Problem 2. Suppose there are 3 men (M) and 3 women (W) with the following endowments of labor: $M_1 = 80$, $M_2 = 90$, $M_3 = 100$, $W_1 = 90$, $W_2 = 100$, $W_3 = 110$. In this game, a man is matched with a woman and they can produce a final good according to the following production function: $F(M, W) = 100 - (M - W)^2$.

(a) Find the optimal match.

(b) Suppose you have not done the previous part. Can you say whether positive assortative matching or negative assortative matching is optimal in this problem.

$M_1 < M_2 < M_3$

$W_2 < W_2 < W_3$

	W_1 90	W_2 100	W_3 110
80 M_1	0	-300	-800
90 M_2	100	0	-300
100 M_3	0	100	0

Optimal $\Leftrightarrow$ Stable

In endowments:

$$F(m,w) = 100 - (m-w)^2 \longleftrightarrow m \simeq w$$

$$\frac{\partial F(m,w)}{\partial m \partial w} = 2 > 0 \quad \Leftrightarrow \Delta_w \Delta_M (F(M,W)) > 0.$$

Positive assortative match in endowments.

$M' > M \qquad \Delta_W \left(F(M',W) - F(M,W)\right) > 0$

$W' > W \qquad F(M',W') - F(M,W') - [F(M',W) - F(M,W)] > 0$

Problem 3. Find the optimal match and a stable allocation in the following:

	Pa	Pb	Pc
H1	5	8	2
H2	7	9	6
H3	2	3	0

$\boxed{1-b}$
$2-c$
$3-a.$
$v \cdot x = 16.$

$q_1, \ t, \ v$

$q_1 + t_a \geq 5$
$q_1 + t_b = 8 \rightarrow t_b = 8 - q_1$
$q_1 + t_c \geq 2.$

$q_2 + t_c = 6 \rightarrow t_c = 6 - q_2$
$q_2 + t_a \geq 7$
$q_2 + t_b \geq 9$

$q_3 + t_a = 2 \rightarrow t_a = 2 - q_3$
$q_3 + t_b \geq 3$
$q_3 + t_c \geq 0$

$q_3 = 0 \rightarrow t_a = 2.$
$q_1 = 4 \rightarrow t_b = 4.$
$q_2 = 5.5 \rightarrow t_c = 0.5.$

(b) People have preferences:

	1	2	3
Pa	\$5	\$7	\$2
Pb	8	9	3
Pc	2	6	0

$b - 1$
$c - 2$
$a - 3$

$q_3 = 0$
For P_c:
$6 - q_2 \geq 0 \rightarrow q_2 \leq 6$
$6 - q_2 \geq 2 - q_1.$

For p_a: $2 \geq 7 - q_2 \longrightarrow q_2 \geq 5.$

$2 \geq 5 - q_1 \longrightarrow q_1 \geq 3.$

For P_b:

$8 - q_1 \geq 9 - q_2$

$\underbrace{8 - q_1}_{= t_1} \geq 3. \longrightarrow q_1 \leq 5.$

$t_1 \geq 9 - q_2.$

$q_2 + t_1 \geq 9.$