ECON 6100 5/14/2021

Section 12

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Problem 1 (2011 June V)**.** Consider an economy in which there is one public good (x) and one private good (y). There are I individuals, indexed $i = 1, \ldots, I$ (with $I \geq 2$). Individual i has an endowment $a_i > 0$ of the private good, and none of the public good. The total endowment of the private good, $(a_1 + \cdots + a_I)$, is denoted by a. The public good can be produced from the private good, using a production function, $h : \mathbb{R}_+ \to \mathbb{R}_+$. Assume that h has the following form: $h(z) = z$ for $z \in \mathbb{R}_+$.

Each individual's consumption set is $\mathbb{R}^2_+$ and consumer i's preferences are represented by a utility function:

$$u_i(x, y_i) = f_i(x) + g_i(y_i) \quad \text{for } (x, y_i) \in \mathbb{R}^2_+$$

$f_i'(a_i) \le f_i(a)$
$< g_i(a_i)$

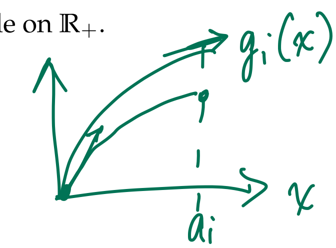
For each $i \in \{1, \ldots, I\}$, the functions f_i and g_i are assumed to satisfy:

(A1) $f_i(0) = 0$; f_i is increasing, strictly concave and continuously differentiable on $\mathbb{R}_+$.

(A1) $g_i(0) = 0$; g_i is increasing, strictly concave and continuously differentiable on $\mathbb{R}_+$.

(A1) $f_i(a) < g_i(a_i)$ and $f_i'(0) > g_i'(a_i)$

$\Sigma_i a_i$



(a) Let $(x, y_1, \ldots, y_I) \gg 0$ be a Pareto Efficient allocation. Show that:

$$\boxed{\sum_{i=1}^{I} \frac{f_i'(x)}{g_i'(y_i)} = 1} = \sum_i \theta_i$$

(b) Let $(c_1, \ldots, c_I)$ be a voluntary contributions equilibrium, with $c_i \in [0, a_i]$ for each $i \in \{1, \ldots, I\}$. The associated allocation $(x, y_1, \ldots, y_I)$ is defined by:

$\forall i$, UMP(i)
MC

$$x = \sum_{i=1}^{I} c_i \quad \text{and} \quad y_i = a_i - c_i \quad \text{for all} \quad i \in \{1, \ldots, I\}$$

(i) Show that we must have $c_i < a_i$ for each $i \in \{1, \ldots, I\}$, and $\sum_{i=1}^{I} c_i > 0$.

(ii) Using (i), show that the allocation $(x, y_1, \ldots, y_I)$, associated with a voluntary contributions equilibrium $(c_1, \ldots, c_I)$, cannot be Pareto Efficient.

(c) Let $(c_1, \ldots, c_I)$ be any voluntary contributions equilibrium, satisfying $(c_1, \ldots, c_I) \gg 0$, with associated allocation $(x, y_1, \ldots, y_I)$. Let $(x', y_1', \ldots, y_I')$ be any Pareto Efficient Allocation satisfying $(x', y_1', \ldots, y_I') \gg 0$. Can $x \geq x'$?

Problem 2 (2009 Aug III)**.** Consider an economy with two consumers, A and B and two assets, 1 and 2. There are three units of asset 1 and three units of asset 2 in the economy. The initial endowment of A at $t = 0$, is given by $(e_1^A, e_2^A) = (2, 1)$, and the initial endowment of B at $t = 0$ is $(e_1^B, e_2^B) = (1, 2)$. The price of asset 1 is q_1, the price of asset 2 is $q_2 = 1$.

At $t = 1$, there are two possible states $S = \{\omega_1, \omega_2\}$, which occur with equal probability. The payoff matrix is given by:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

Consumers are both expected utility maximizer with utility for state-contingent wealth x given by

$$u^A(x) = 5\ln x + 2$$
$$u^B(x) = 13x$$

$EU = \frac{1}{2}\ln(x_1) + \frac{1}{2}\ln(x_2)$

$= \sqrt{x_1 x_2}$

$EU = x_1 + x_2$

(a) At $t = 0$, the two consumers choose portfolios of assets so as to maximize their expected utility of state-contingent consumption. State the optimization problems of the two consumers at $t = 0$.

$x_2 = \infty$

(a′) Suppose $q_1 = \frac{5}{2}$. Draw the budget constraint of consumer A. What is the optimal choice of consumption in state ω_2 for this consumer? Derive the set of values of q_1 for which the budget sets of both consumers are bounded.

(b) For the set of values of q_1 derived in part (a′), solve the optimization problems of both consumers. Set up the conditions for a market equilibrium and derive the equilibrium consumption and asset prices. Illustrate the equilibrium in an Edgeworth box.

(c) Which of the two consumers is fully insured in equilibrium? Show that this consumers will be fully insured in equilibrium for any distribution of initial endowments such that: $e_1^A > 0$, $e_2^A > 0$, $e_1^B > 0$, $e_2^B > 0$, $e_1^A + e_1^B = 3$, $e_2^A + e_2^B = 3$, and $e_1^A + e_2^A \leq 3$.

Let A endow $= a_1, a_2$. ; $a_1 + a_2 \leq 3$

B endow $= 3 - a_1, 3 - a_2$.

(d) New research has uncovered a third state, ω_3 which can occur at $t = 1$ with probability 0.2. States ω_1 and ω_2 are still considered to be equally probable. The payoff matrix is now

$U_1 = (x_1 x_2 x_3)^{1/3}$ $(0, 1, 0)$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Rank$(A) = 2$.

$|S| = 3$.

Is it possible to determine whether the equilibrium of this economy is Pareto-optimal without actually computing it?

(d′) How would your answer to (d) change if there were a third asset and the payoff matrix, for some $r \in \mathbb{R}^1_+$, is now:

$\forall r < 3$

$$\begin{array}{cccc} & + & & = \\ A = & \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & r \end{bmatrix} \end{array}$$

$\forall r$?

$r = 3$

$3 - r.$

> 0

① a) PO allocation $\Longrightarrow$ solve social planner's problem w/ Pareto weight $\theta_1, \dots, \theta_I : \sum_i \theta_i = 1$.

$$\max_{x, \vec{y}} \sum_i \theta_i \left[f_i(x) + g_i(y_i) \right]$$

s.t.

$$x + \sum_i y_i \leq a \quad \text{---} (\lambda)$$

$$x \geq 0 \quad \text{---} (\mu)$$

$$y_i \geq 0 \quad \text{---} (\mu_i)$$

KT opt:

FOCs: (x) : $\sum_i \theta_i f_i'(x) - \lambda + \mu = 0$

$\forall i$ (y_i) : $\theta_i g_i'(y_i) - \lambda + \mu_i = 0$.

Ineq. const.

$$(a - x - \sum_i y_i)\lambda = 0, \quad \mu x = 0, \quad \mu_i y_i = 0, \forall i$$

$$\lambda, \mu, \mu_i \geq 0$$

We know, $y_i > 0 \Rightarrow \mu_i = 0$, $\forall i$

$x > 0 \Rightarrow \mu = 0$, $\forall i$.

FOCs: $\sum_i \theta_i f_i'(x) = \lambda = \theta_i g_i'(y_i)$ ---- $\underline{\underline{\forall i}}$

$$\frac{\sum_i \theta_i f_i'(x)}{\lambda} = 1$$

$$= \frac{\sum_i \theta_i f_i'(x)}{\theta_j g_j'(y_j)} = 1. \longrightarrow \text{Take sum } \underline{\text{outside!}}$$

$$\frac{3+5}{2} = \frac{3}{2} + \frac{5}{2}$$

$\longrightarrow \theta_k g_k'(y_k), \forall k$.

(b) UMP(i): $\max_{c_i} \; f_i\left(\sum_{j\neq i} c_j + c_i\right) + g_i(a_i - c_i)$

s.t. $\quad 0 \underset{(\mu_i)}{\le} c_i \underset{(\lambda_i)}{\le} a_i$

KT opt: FOC: $f_i'\left(\sum_{j\neq i} c_j + c_i\right) - g_i'(a_i - c_i) - \lambda_i + \mu_i = 0$.

Ineq. ; C.S ; non-neg.

(1) "$c_i < a_i, \forall i$"

Suppose $c_i = a_i \longrightarrow$ By CS $\rightarrow \mu_i = 0$; $\lambda_i \geq 0$

FOC: $\underbrace{f_i'\left(\sum_{j\neq i} c_j + c_i\right) - g_i'(0)}_{\geq 0} - \underset{\geq 0}{\lambda_i} = 0$.

$$f_i'\left(\sum_{j\neq i} c_j + c_i\right) \geq g_i'(0).$$

$\Longrightarrow$

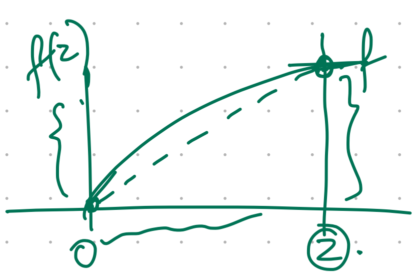
$$\frac{f_i(a)}{\sum_{j\neq i} c_j + a_i} \geq \frac{f_i\left(\sum_{j\neq i} c_j + a_i\right)}{\sum_{j\neq i} c_j + a_i} > f_i'\left(\sum_{j\neq i} c_j + a_i\right) \geq g_i'(0) > \frac{g_i(a_i)}{a_i}$$

$a = \sum a_i \geq \sum_{j\neq i} c_j + a_i$

Concavity $f_i(0) = 0$

Concavity $g(0) = 0$

$$f(z) - f(0) > z \cdot f'(z).$$



$$\frac{f_i(a)}{g_i(a_i)} > \frac{\sum_{j\neq i} g + a_i}{\underbrace{a_i}} \geq 1 .$$

$$f_i(a) > g_i(a_i) \rightarrow \underline{\underline{\text{Contradiction}}}$$

" $\sum_i c_i > 0$ ":

↳ Suppose $c_i = 0, \forall i \Rightarrow \lambda_i = 0, \forall i$.

$$\text{FOC} \Rightarrow \underbrace{f_i'(0) - g_i'(a_i)}_{>0} + \underbrace{\mu_i}_{\geq 0} = 0 .$$

$\underline{\underline{\text{Contradiction}}}$

(ii) From i: $c_i < a_i, \forall i \Rightarrow \lambda_i = 0$

Some i: $c_i > 0 \Rightarrow \mu_i = 0$.
(atleast 1).

↳ For these i FOC: $f_i'(\sum_{j\neq i} g + c_i) - g_i'(c_i) = \underline{0}$

For others FOC: $f_i'(\quad) - g_i'(a_i) + \mu_i = 0$.

$$\Rightarrow \alpha \frac{f_i'(\cdot)}{g_i'(a_i)} = 1 - \frac{\mu_i}{g_i'(a_i)} \leq 1$$

In CE: $\sum_i \frac{f_i'(\sum_i c_i)}{g_i'(a_i - c_i)} = \underbrace{\sum_{i: c_i = 0} {}^{u}}_{>0} + \underbrace{\sum_{i: c_i > 0} {}^{u}}_{\leq 1}$.

$\underline{\underline{> 1}}$. (In PO $= 1$) $\rightarrow$ (a).

Contradiction.

(c) Shown above.

VE : $\sum_i \frac{f_i'}{g_i'} > 1$.

PO : $\quad {}^{u} \quad = 1$.

$\exists i$: $\frac{f_i'(x)}{g_i'(y_i)} > \frac{f_i'(x')}{g_i'(y_i')} = 1$.

$\vdots$

+ Concavity.

$\frac{f_i'(x)}{f_i'(x')} > 1. \Rightarrow \underline{\underline{x' > x}}$.

$\boxed{2}$ a) $q_1 = 2.5$

$$UMP(A) = \max_{x^A, z^A} \quad \ln(x_1^A) + \ln(x_1^B)$$

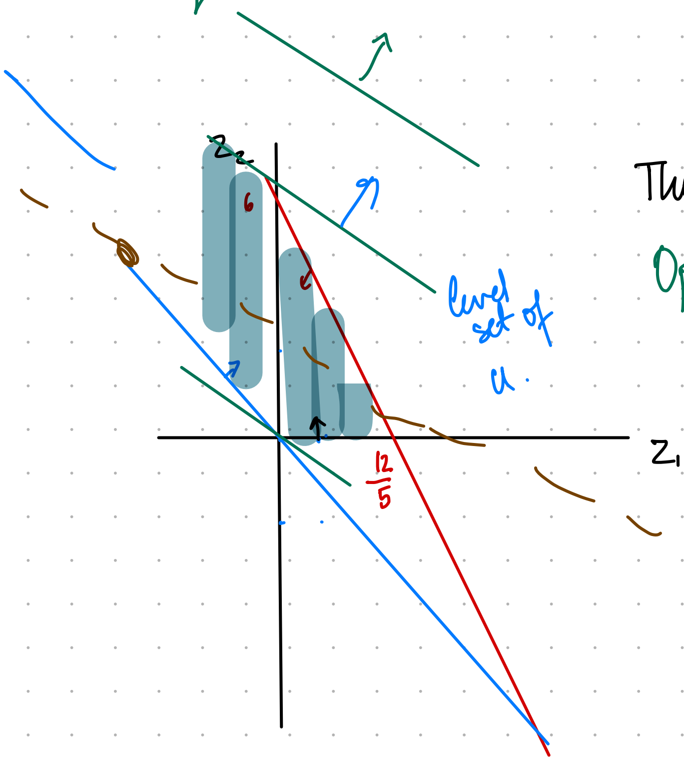
$$\text{s.t.} \quad q_1 z_1^A + z_2^A \leq 2q_1 + q_2$$

$$0 \leq x_1^A \leq 2z_1^A + z_2^A$$

$$0 \leq x_2^A \leq z_2^A$$

$$x^A \geq 0$$

$BC_A(z^A)$ when $q_1 = 2.5$, $q_2 = 1$ $= \left\{ z^A : \underline{2.5 z_1^A + z_2^A \leq 6}; \ \underline{2z_1^A + z_2^A \geq 0}, \ z_2^A \geq 0 \right\}$



Thus, the BC is unbounded!

Optimal choice of $z_2 = \infty$

slope of brown $> -\frac{1}{2}$.

$$-1 \frac{1}{q_1} > +\frac{1}{2}$$

$$\boxed{q_1 < 2}$$

In regd. of bdd of both agents constraints

b) UMP A: $\max \ \ln x_1^A + \ln x_2^A \quad \longleftrightarrow$ affine transform

$$\text{s.t.} \quad q_1 z_1^A + z_2^A \leq 2q_1 + 1 \quad \rightarrow z_2^A = 2q_1 + 1 - q_1 z_1^A.$$
$$0 \leq x_1^A \leq 2z_1^A + z_2^A$$
$$0 \leq x_2^A \leq z_2^A$$

$$\max \ \ln\left(4 + \frac{2}{q_1} + \left(1 - \frac{2}{q_1}\right) z_2^A\right) + \ln z_2^A$$

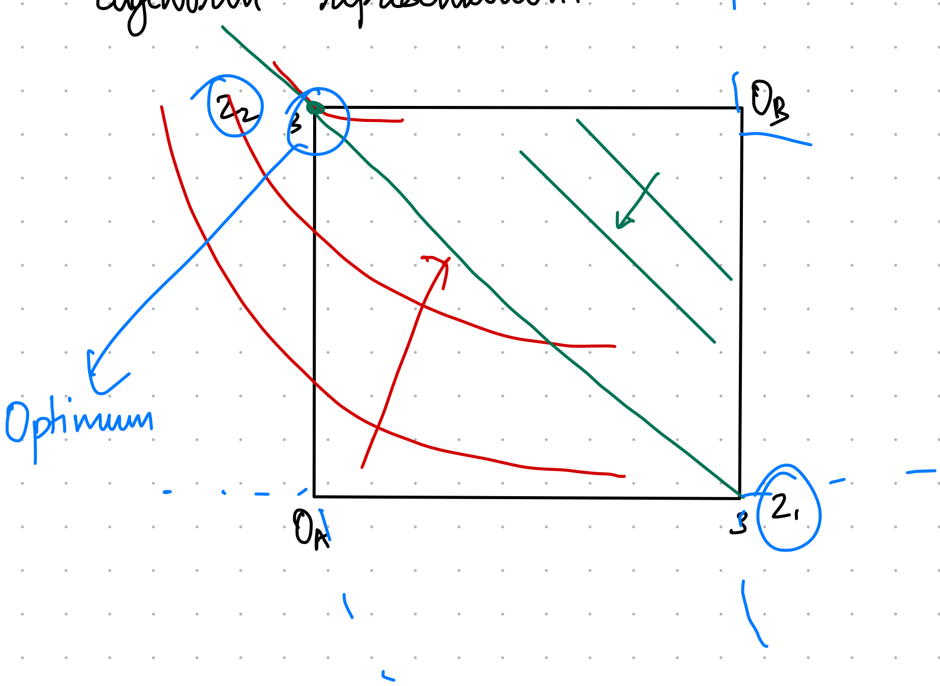
$$z_2^{A*} = \frac{2q_1+1}{2-q_1} \quad ; \quad z_1^{A*} = \frac{(2q_1+1)(1-q_1)}{q_1(2-q_1)}$$

$$x_2^{A*} = z_2^{A*} \quad \text{and} \quad x_1^{A*} = 2z_1^{A*} + z_2^{A*}$$

Do the same for B to compute z^{B*}. (Linear system with $q_1 < 2$)
Then, use market clearing to compute q_1:

$$q_1^* = 1, \ z_1^{A*} = 0, \ z_2^{A*} = 3 \ ; \ x_1^{A*} = x_2^{A*} = 3$$
$$z_1^{B*} = 3 \ ; \ z_2^{B*} = 0 \ ; \ x_1^{B*} = 6 \ ; \ x_2^{B*} = 0.$$

Edgeworth representation:



$$U_1 = \ln(2z_1 + z_2) + \ln(z_2)$$
$$U_2 = 2z_1 + z_2 + z_2$$
$$= 2(z_1 + z_2).$$