ECON 6100	02/19/2021

Section 1

Lecturer: Larry Blume	*TA: Abhi Ananth*

* These notes develop Fikri Pitsuwan's notes from 2017.

Logistics

- OH: Thus 4-6 pm
- Material available at: https://abhiananthecon.github.io/teaching/
- Same link for office hours and sections
- Please email me with subject header 6100 to be a part of the mailing list
- Thu 6pm deadline for topic suggestions
- Questions?

Today we will look at:

1. Farka's lemma
2. Canonical and standard form
3. Vertex theorem

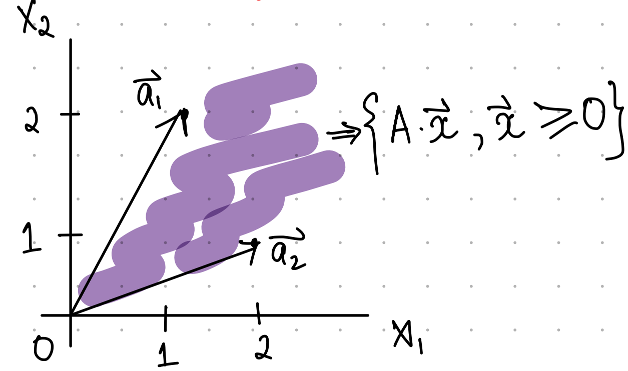
1 Review

Let's start with Farka's lemma. It states that for any $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, **exactly one** of the following **will hold**:

- There is some $x \in \mathbb{R}^n$ satisfying $x \geq 0$ and $Ax = b$.
- There is some $y \in \mathbb{R}^m$ satisfying $yA \geq 0$ and $yb < 0$.

Farka's Lemma:

Suppose $A = \begin{array}{cc} \tilde{a}_1 & \\ \left[\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}\right] & \begin{matrix} \vec{a}_1 \\ \vec{a}_2 \end{matrix} \end{array}$; $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



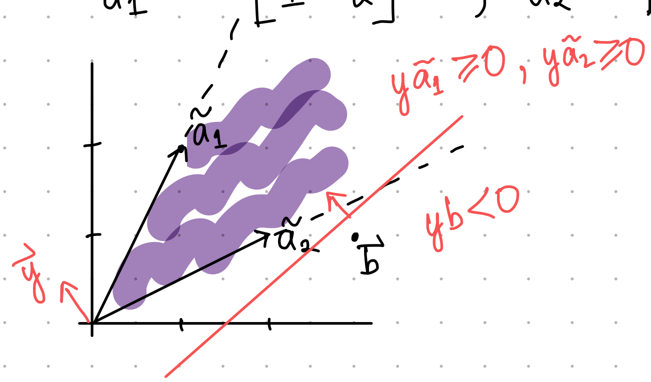
If $\vec{b}$ lies in purple area, $A\vec{x} = b$, $\vec{x} \geq \vec{0}$ has a solution. Eg $\vec{b} = (1,1) \longrightarrow x^*(\vec{b}) = (3/5, 1/5)$

Else, it has no solution. Eg $\vec{b} = (3,1)$.

Farka's lemma says that when $\{\vec{x} \geq 0 : A\vec{x} = b\} = \phi$, then

$$\{\vec{y} : \vec{y}A \geq \vec{0}, \vec{y}b < 0\} \neq \phi.$$

$\tilde{a}_1 = [1 \;\; 2]$; $\tilde{a}_2 = [2 \;\; 1]$ $\quad$ $\tilde{a}_1$: 1st column vector in A.



Why it matters?

- Neat application of the separating hyperplane theorem
- Easy to verify criterion for feasibility of a linear program

A linear program can be written in *canonical form* as

$$\begin{aligned} v_p(b) &= \max c \cdot x \\ \text{s.t. } Ax &\le b \\ x &\ge 0 \end{aligned}$$

where $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, A is an $m \times n$ matrix, $b \in \mathbb{R}^m$. Any linear program can also be written in *standard form* as

$$\begin{aligned} v_p(b) &= \max c \cdot x \\ \text{s.t. } Ax &= b \\ x &\ge 0 \end{aligned}$$

Given an inequality constraint $2x_1 + 3x_2 \le 5$ and $x_1, x_2 \ge 0$, we can introduce a slack variable $z_1 \ge 0$, so that the constraint becomes $2x_1 + 3x_2 + z_1 = 5$. Given an equality constraint $x_1 + 2x_2 = 3$, we can express this as $x_1 + 2x_2 \le 3$ and $-x_1 - 2x_2 \le -3$. A linear program with no non-negativity constraint can be dealt with by expressing $x = y - z$ with $y \ge 0$ and $z \ge 0$.

Here are some important definitions in linear programming.

Definition 1. Any $x \in \mathbb{R}^n$ is called a *solution*.

Definition 2. For a linear program in canonical form, $C = \{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}$ is called the *constraint set* or the *feasible set*. Any $x \in C$ is called a *feasible solution*.

Definition 3. A vector x that actually solves the linear program, i.e., $x \in C$ and $c \cdot x \ge c \cdot x'$ for all $x' \in C$ is called an *optimal solution*.

Definition 4. A vector $x \in C$ is a vertex of C if and only if there is no $y \ne 0$ such that $x + y$ and $x - y$ are both in C.

Theorem (Vertex Theorem). *For a linear program in standard form with feasible solutions, a vertex exists and if $v_p(b) < \infty$ and $x \in C$, then there is a vertex x' such that $c \cdot x' \ge c \cdot x$.*

Notes

A good reference on linear programming is *Introduction to Linear Optimization* by Bertsimas and Tsitsiklis.

2 Problems

Problem 1. Consider the following linear program

$$\max 2x_1 + x_2$$
$$\text{s.t. } x_1 + x_2 \le 1$$
$$x_1 \ge 0,\ x_2 \ge 0$$

(a) Express the linear program in canonical form and draw the constraint set and solve the problem graphically.

(b) Express the linear program in standard form and draw the constraint set.

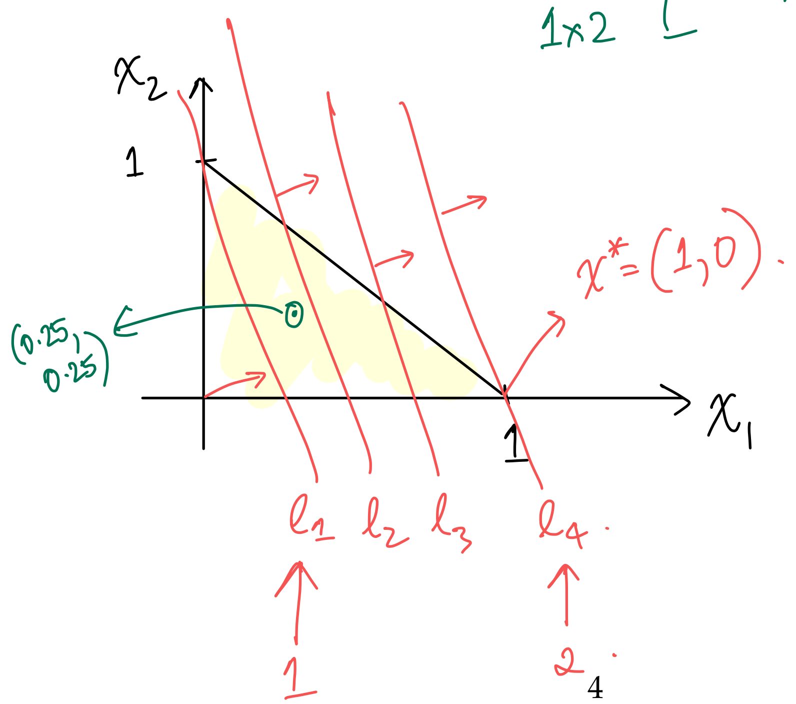
(c) Verify that the vertex theorem applies. Use the vertex theorem to find an optimal solution of the linear program.

(a) Canonical form:

$\underset{2\times 1}{x} = (x_1, x_2)$

$$\max_{x} \ \vec{c}\cdot x$$
$$\text{s.t. } Ax \le \vec{b},\ x \ge 0.$$

$\vec{c} = (2, 1)$; $\underset{1\times 2}{A} = [1, 1]$; $\underset{1\times 1}{b} = 1.$



— = Feasible

$c\cdot x = l_1$ >

$c\cdot x. = l_2$ >

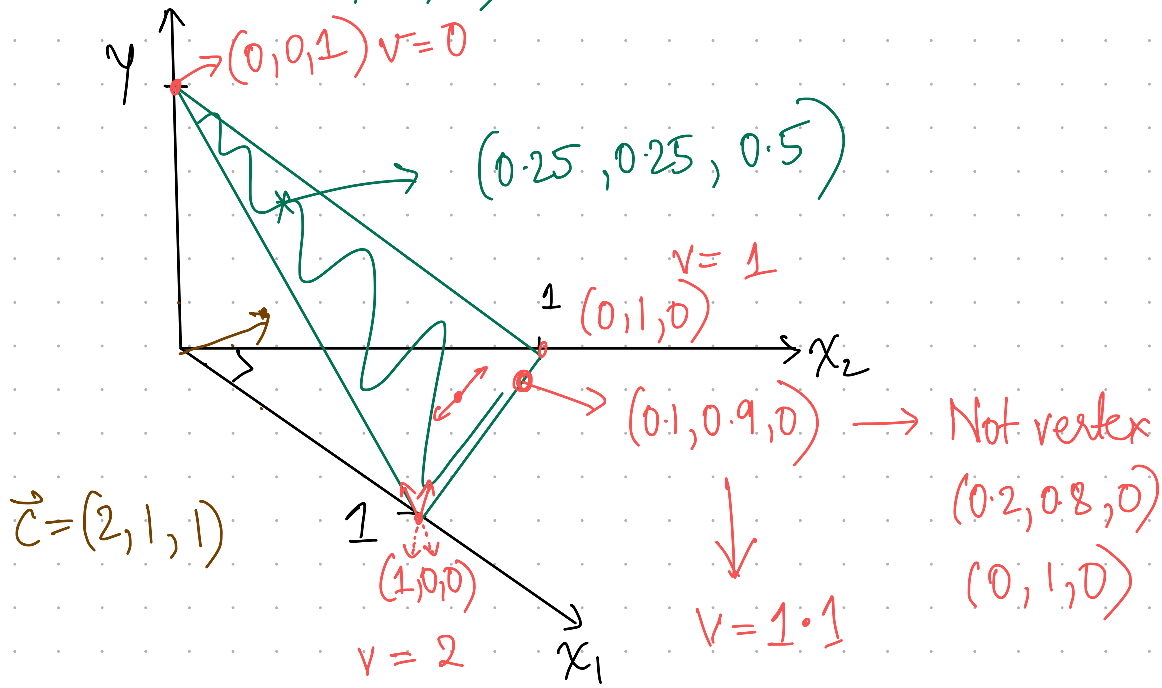
$c\cdot x = l_3$

(b) $\max_{z} \; 2x_1 + x_2 + y \cdot 0$ $z = (x_1, x_2, y)$

s.t. $x_1 + x_2 + y = 1$.

$x_1 \geq 0, \; x_2 \geq 0, \; y \geq 0$

$\tilde{c} = (2, 1, 0)$; $\tilde{A} = [1, 1, 1]$; $\tilde{b} = 1$.



$2x_1 + x_2 = 2(y_1 - z_1) + (y_2 - z_2)$

$= 2y_1 - 2z_1 + y_2 - z_2$

$x_1 = y_1 - z_1$

$x_2 = y_2 - z_2$

max $(2, -2, 1, -1, 0, 0)\begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ u \\ v \end{bmatrix}$

s.t. $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & -2 & 2 & 0 & 1 \end{bmatrix} (\cdot) \le \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y_1, y_2, z_1, z_2, u, v \ge 0$.

Problem 2. Consider the following linear program

$x_1 - 2x_2 + v = 1$

$$\begin{aligned} \max\ & 2x_1 + x_2 \\ \text{s. t. }\ & x_1 + x_2 \le 1 \\ & 2x_2 - x_1 \ge -1 \end{aligned}$$

$\boxed{x_1 - 2x_2 \le 1}$

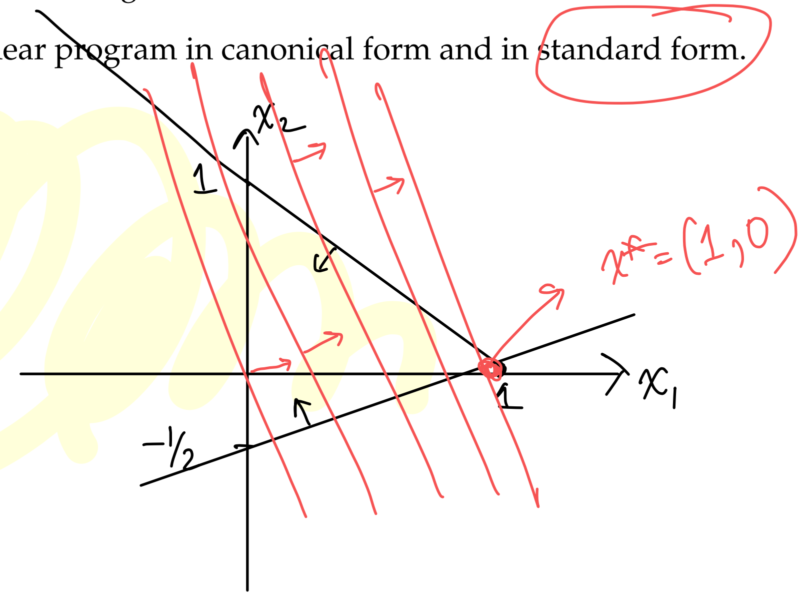
$y_1 - z_1 - 2y_2 + 2z_2 \le 1$

(a) Draw the constraint set as given and solve the problem graphically.

(b) Solve the problem using the Kuhn-Tucker formulation

(c) Express the linear program in canonical form and in standard form.

(a)



(b) ~~Largane~~: Lagrangian:

$$L = 2x_1 + x_2 + \overset{\ge 0}{\lambda_1}(1 - x_1 - x_2) + \overset{\ge 0}{\lambda_2}(1 - x_1 + 2x_2)$$

KT conditions:

A FOC: (x_1): $2 - \lambda_1 - \lambda_2 = 0 \rightarrow \lambda_2 = 2 - \lambda_1$

B (x_2): $1 - \lambda_1 + 2\lambda_2 = 0 \rightarrow \lambda_2 = \dfrac{\lambda_1 - 1}{2}$.

C C.S. $\lambda_1(1 - x_1 - x_2) = 0$

D $\lambda_2(1 - x_1 + 2x_2) = 0$

E Ineq: $x_1 + x_2 \le 1$

F $x_1 - 2x_2 \le 1$.

G Lag: $\lambda_1, \lambda_2 \geq 0$.

Cases:

$\lambda_1 = 0 \xrightarrow{A} \lambda_2 = 2 \xrightarrow{D} x_1 - 2x_2 = 1$.

$\lambda_1 = 0 \xrightarrow{B} \lambda_2 = -1/2 \longrightarrow$ violates G.

$\downarrow$

Ruled out.

$\lambda_2 = 0 \xrightarrow{A} \lambda_1 = 2$

$\lambda_2 = 0 \xrightarrow{B} \lambda_1 = 1$

$\Big\}\ 1 \neq 2 \longrightarrow$ violates math!

$\downarrow$

Ruled out.

In any optima, both constraints bind:

$$2x_1 + \cancel{2x_1} = 2$$

$$x_1 - \cancel{2x_2} = 1.$$

$$3x_1 = 3 \rightarrow x_1^* = 1; \quad x_2^* = 1 - 1 = 0.$$

(c) $\max_x c \cdot x$ s.t. $Ax = b$; $x \geq 0$.

Problem 3. Consider a utility maximization problem with $u(x) = \sum_{i=1}^{n} \alpha_i x_i$, where $\alpha_i > 0$ for all i.

(a) Express the problem as a linear program in canonical form. What is the feasible set? What are c, A, and b?

(b) Solve the UMP for $n = 2$ using the Kuhn-Tucker formulation with $\alpha_1 = 3$, $\alpha_2 = 2$, $p_1 = 3$, $p_2 = 1$, $w = 3$. Verify your solution graphically.

(a) $$\max_{x} \ \alpha_1 x_1 + \dots + \alpha_n x_n$$

$$\text{s.t.} \quad p \cdot x \leq w.$$

$$x \geq 0.$$

$$c = \vec{\alpha} \ ; \ A = p \ ; \ b = w$$

(b) $$\max_{x} \ 3x_1 + 2x_2$$

$$\text{s.t.} \quad 3x_1 + 1x_2 \leq 3$$

$$\{x_1, x_2 \geq 0\}$$

KT conditions: $\mathcal{L} = 3x_1 + 2x_2 + \lambda_1(3 - 3x_1 - x_2) + \lambda_2 x_1 + \lambda_3 x_2.$

FOCs: (x_1): $3 - 3\lambda_1 + \lambda_2 = 0 \longrightarrow \lambda_2 = 3\lambda_1 - 3$

(x_2): $2 - \lambda_1 + \lambda_3 = 0 \longrightarrow \lambda_3 = \lambda_1 - 2$

CS: $\lambda_1(3 - 3x_1 - x_2) = 0$

$\lambda_2 x_1 = 0$

$\lambda_3 x_2 = 0$

Ineq: + Lag.

Case I: $\begin{cases} \lambda_1 = 0 \Rightarrow \lambda_2 = -3, \lambda_3 = -2. \\ \hookrightarrow \text{Ruled out.} \end{cases}$

$\begin{cases} \lambda_2 = 0 \Rightarrow \lambda_1 = 1, \lambda_3 = -1. < 0 \\ \hookrightarrow \text{Ruled out.} \end{cases}$

$\lambda_3 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3.$

$\Rightarrow \lambda_2 \neq 0 \Rightarrow x_1 = 0.$

$\lambda_1 \neq 0 \Rightarrow 3\cancel{x_1} + x_2 = 3$
$\quad\quad\quad\quad\quad\quad 0$

$\Rightarrow \underline{\underline{x_2 = 3}}$