ECON 6100	02/26/2021

Section 2

Lecturer: Larry Blume	*TA: Abhi Ananth*

* These notes develop Fikri Pitsuwan's notes from 2017.

1 Review

The feasible set of a linear program in standard form is $C = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$. For $x \in \mathbb{R}^n$, define $supp(x) = \{j : x_j > 0\}$, the set of coordinates such that x has strictly positive component.

Definition 1. A feasible solution $x \in C$ is *basic* if the set $\{A^j : j \in supp(x)\}$ is linearly independent.

Note that since A is $m \times n$ and without loss of generality we can assume that A has full row rank, i.e., $\text{rank}(A) = m$, for $x \in C$ to be basic, x needs to pick out m linearly independent columns of A.

Theorem. *$x \in C$ is basic if and only if it is a vertex.*

Example 1. Consider

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \end{bmatrix}$$

For $x = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}^T$, $x \in C$ and $supp(x) = \{1, 2\}$. Since $\{A^1, A^2\}$ is not linearly independent, x is not basic and thus not a vertex. The basic feasible solutions are $x = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $x = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

Theorem (FTLP)**.** *If a linear program in standard form has an optimal solution, then it has a basic optimal solution.*

Given a linear program in canonical form

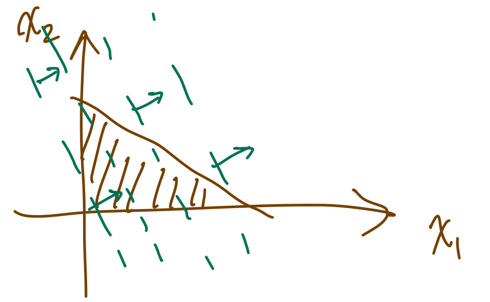
$$\begin{aligned} v_p(b) = \max c \cdot x \\ \text{s.t. } Ax \leq b \\ x \geq 0 \end{aligned}$$

The *dual* linear program is

$$\begin{aligned} v_D(c) = \min y \cdot b \\ \text{s.t. } yA \geq c \\ y \geq 0 \end{aligned}$$

A very useful theorem from the study of duality is the complementary slackness theorem.

Theorem. *If x^* and y^* are feasible for the primal and dual, then they are optimal if and only if $y^*(b - Ax^*) = 0$ and $(y^*A - c)x^* = 0$*



Example 2. Consider the linear program from section 1

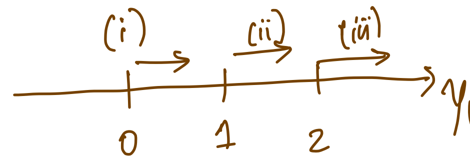
$$\begin{aligned} \max\ & c \cdot x \\ \text{s.t}\ & Ax \le b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} &\max 2x_1 + x_2 \\ \text{s. t.}\ & x_1 + x_2 \le 1 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

The dual linear program is

$$\begin{aligned} \min\ & by \\ \text{s.t.}\ & yA \geq c \\ & y \geq 0 \end{aligned}$$

$$\begin{aligned} &\min y_1 \\ \text{s. t.}\ & y_1 \ge 2 \text{ (iii)} \quad y_1 \ge 1 \text{ (ii)} \\ & y_1 \ge 0 - \text{i} \end{aligned}$$



Clearly, the solution to the dual is $y_1^* = 2$, so an optimal solution of the primal must satisfy $x_1^* + x_2^* = 1$. Now, we also have $(y_1^* - 2)x_1^* + (y_1^* - 1)x_2^* = 0$, which implies that $x_1^* = 1$ and $x_2^* = 0$.

$$y^*(b - Ax^*) = 0.$$

$$\underset{\substack{\neq \\ 0}}{y^*}\ \underbrace{(1 - x_1^* - x_2^*)}_{= 0} = 0.$$

$$x_1^* = 1 - x_2^*.$$

$$(\overset{=2}{y^*}\ [1\ \ 1]x^* - 2x_1^* - x_2^*) = 0.$$

$$\cancel{2x_1^*} + 2x_2^* - \cancel{2x_1^*} - x_2^* = 0$$

$$x_2^* = 0.$$

$$x_1^* = 1 \,//$$

2 Problems

Problem 1. Consider the (primal) linear program

$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

$$\begin{aligned} &\max x_1 + x_2 \\ \text{s. t. } & x_1 + 2x_2 \leq 6 \\ & x_1 - x_2 \leq 3 \\ & x_1 \geq 0, \ x_2 \geq 0 \end{aligned}$$

$x_1 + 2x_2 + x_3 = 6$
$x_1 - x_2 + x_4 = 3.$
$x_1, x_2, x_3, x_4 \geq 0$

$yA = [y_1 + y_2, \ 2y_1 - y_2]^T$

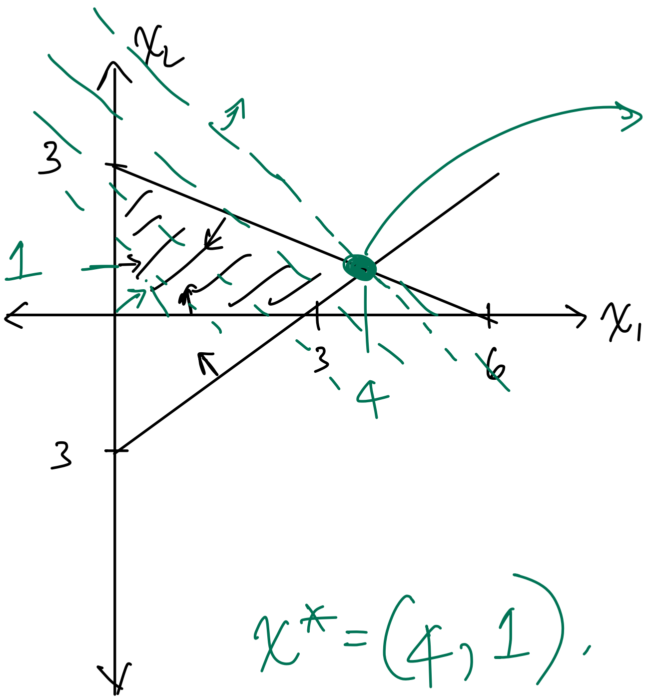
(a) Draw the constraint set and solve graphically.

(b) Write the problem in standard form.

(c) State and solve the dual problem.

(d) Verify that the values coincide and that the complementary slackness conditions hold.

(a)



$x_1 + 2x_2 = 6$
$-\quad x_1 \underset{(+)}{-} x_2 = 3 \quad -$

$3x_2 = 3$

$x_2 = 1.$

$x_1 = 4.$

$x^* = (4, 1).$

(b) Std. form: $\quad c, A, b, x.$?

$\max \ c \cdot x$
s.t. $Ax = b$
$x \geq 0$

$x = (x_1, x_2, x_3, x_4)$
$c = (1, 1, 0, 0)$
$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$

$b = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

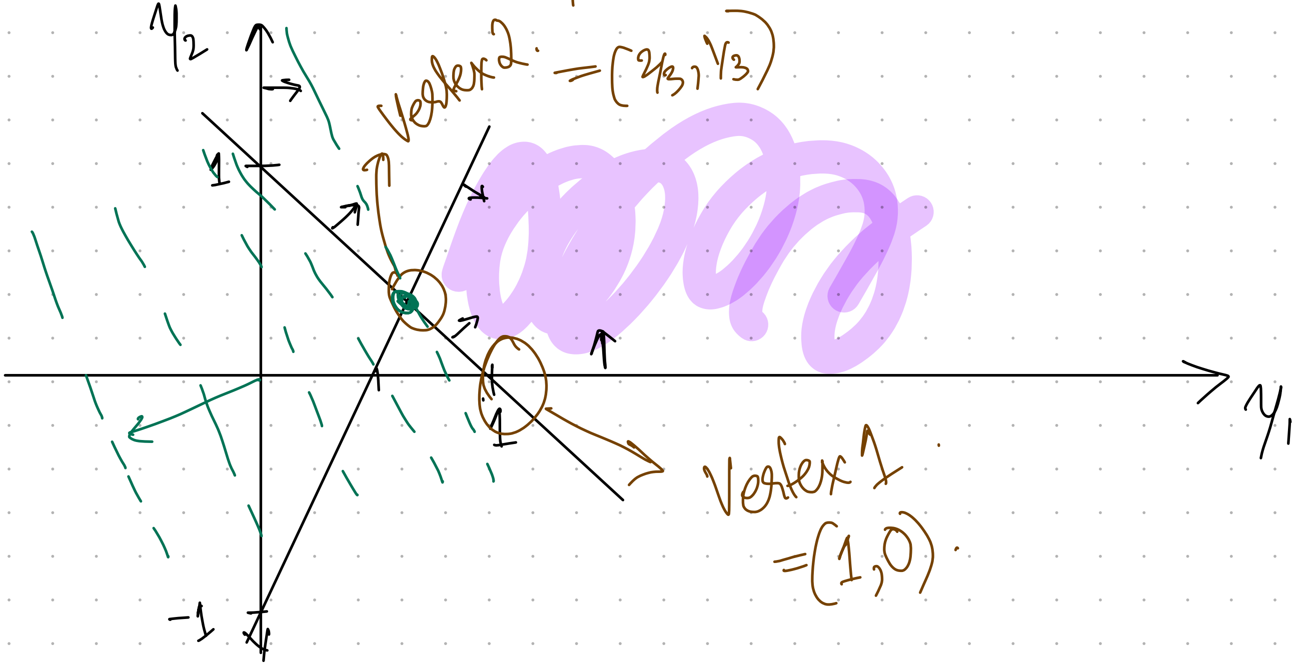
(c) Dual:

$$\min_{y} \quad 6y_1 + 3y_1 \qquad (y \cdot b).$$

$$\text{s.t.} \quad y_1 + y_2 \geq 1 \qquad (y \cdot A \geq c)$$

$$2y_1 - y_2 \geq 1$$

$$y_1, y_2 \geq 0$$



$$y_2 = 1 - y_1$$

$$2y_1 - 1 + y_1 = 1.$$

$$y_1 = 2/3.$$

$$\text{Value}_D(V1) = 6y_1 + 3y_2 = 6.$$

$$\text{Value}_D(V2) = 6\left(\tfrac{2}{3}\right) + 3\left(\tfrac{1}{3}\right) = 5$$

(d) $y^*[b - Ax^*] = 0$.

$[y^*A - c]x^* = 0$.

$$\begin{pmatrix} 2/3 & 1/3 \end{pmatrix} \left[\cancel{\begin{pmatrix} 6 \\ 3 \end{pmatrix}} - \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix}} \right]$$

$$\begin{pmatrix} 4+2 \\ 4-1 \end{pmatrix} = \cancel{\begin{pmatrix} 6 \\ 3 \end{pmatrix}}$$

$$\begin{pmatrix} 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \underline{\underline{0}}.$$

Problem 2. Consider the following linear program

$$
\begin{aligned}
v_P(b) = &\max x_1 + 2x_2 \\
\text{s.t. } & x_1 + x_2 \le 4 \\
& x_1 + 3x_2 \le b \\
& x_1 \ge 0,\ x_2 \ge 0
\end{aligned}
$$

$\longrightarrow$ moving around.

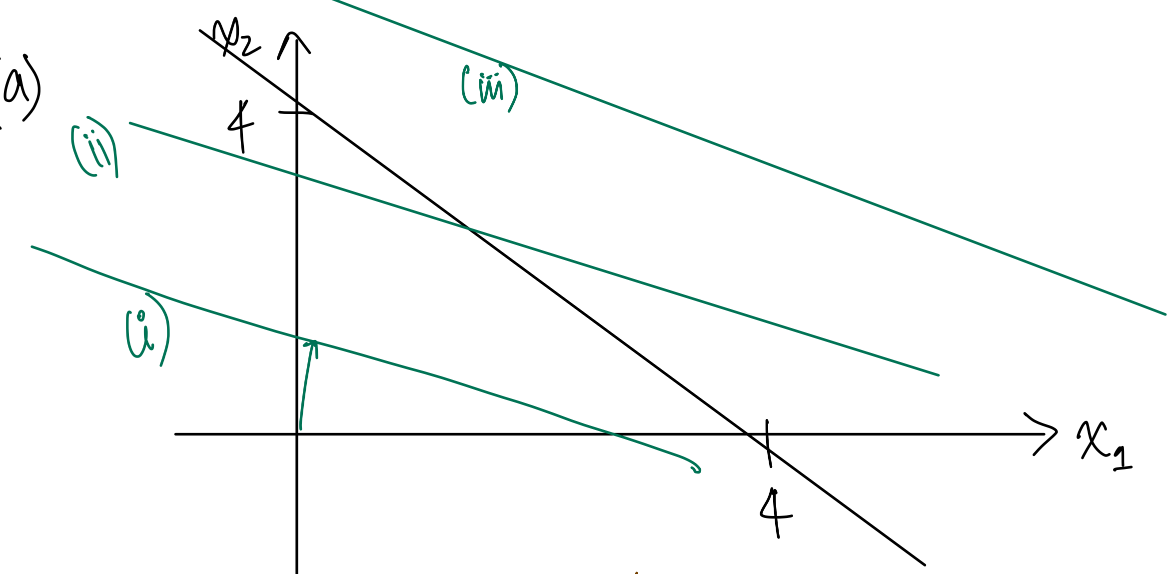
(a) Draw the constraint set.

(b) Solve the problem and plot $v_p(b)$.

(c) State and solve the dual problem. How does the solution of the dual problem depend on b?

(d) Let $b = 6$, verify the complementary slackness conditions.

(a)



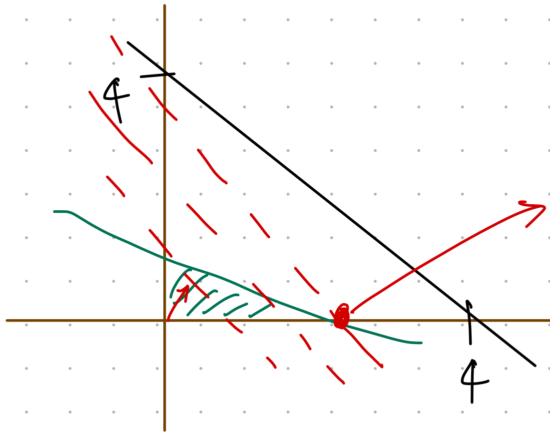
(i) $x_1 + 3x_2 \le b$ | $x_2 = 0$ $\to x_1 \le b \le 4$.

$b \le 4$.

(iii) $x_1 + 3x_2 \le b$ | $x_1 = 0 \to x_2 = b/3$. $b/3 > 4$. $\Rightarrow b > 12$.

(ii) $4 < b \le 12$

In case (i)



$(x_1^* = b\ ,\ x_2^* = 0)$.

$V_p = b$

Case (ii)

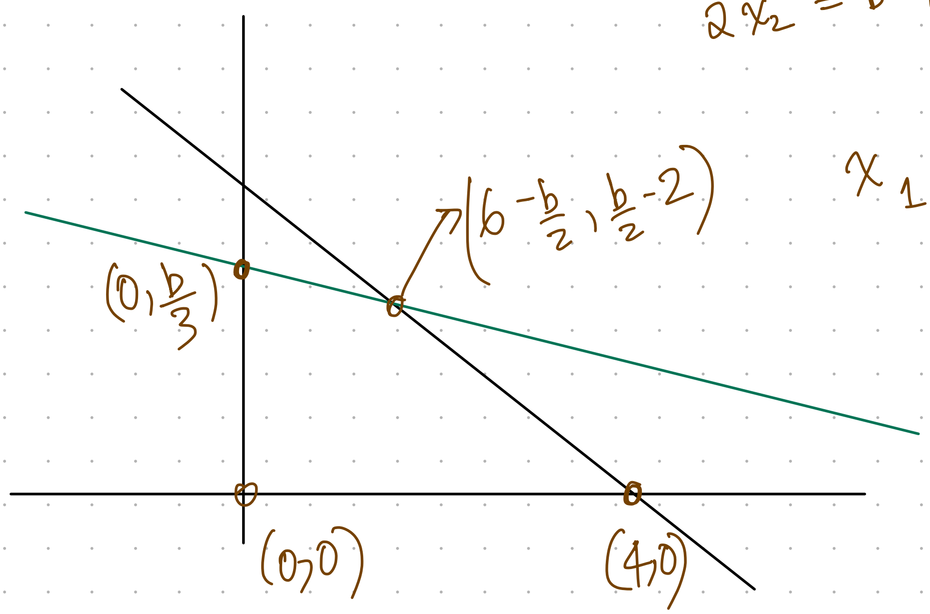
$x_1 + x_2 = 4$

$x_1 + 3x_2 = b$

$2x_2 = b - 4 \Rightarrow x_2 = \frac{b}{2} - 2$

$x_1 = 4 - \frac{b}{2} + 2$

$= 6 - \frac{b}{2}$.



$V_p(0,0) = 0$

$V_p(4,0) = 4$

$V_p\left(0, \frac{b}{3}\right) = \frac{2}{3}b \xrightarrow{4 \le b \le 12} \frac{8}{3}, 8$

$V_p\left(6 - \frac{b}{2}, \frac{b}{2} - 2\right) = 6 - \frac{b}{2} + b - 4$.

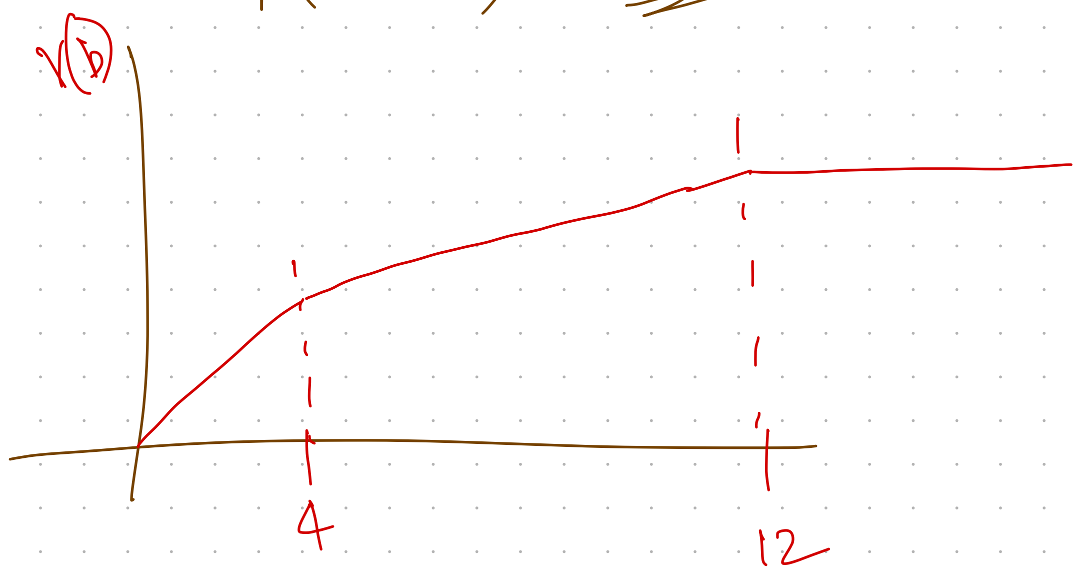
$= 2 + b/2 \rightarrow 4, 8$

When $4 \le b \le 12$.

$\Rightarrow V_P\left(6-\frac{b}{2}, \frac{b}{2}-2\right)$ is highest.

When $b > 12$.

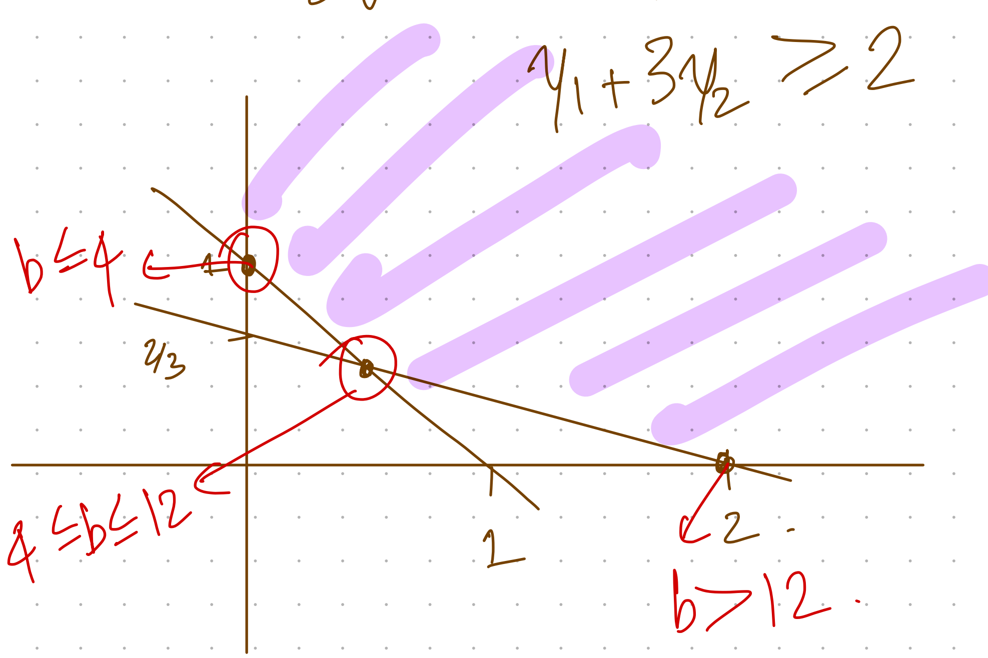
$\rightarrow V_P(0,4) = \underline{\underline{8}}$.



(c) Dual: $\min_y \; 4y_1 + b y_2$

s.t. $y_1 + y_2 \ge 1$.

$y_1 + 3y_2 \ge 2$, $y_1, y_2 \ge 0$



$(4, b)$.

1 holds $\Rightarrow$ 2 doesn't.
2 holds $\Rightarrow$ 1 doesn't.

Problem 3. Prove Gordon's Lemma: Let $A \in \mathbb{R}^{n\times m}$, then exactly one of the two alternatives is true:

1. $\exists x \in \mathbb{R}^n, x \neq 0, x \geq 0$ such that $Ax = 0$
2. $\exists y \in \mathbb{R}^m$ such that $yA >> 0$

Farka's lemma: Exactly one of the following holds:

i - $Ax = b$ for some $x \geq 0$

ii - $yA \geq 0$, $yb < 0$ for some y.

1 holds iff 2 doesn't
$\Longleftrightarrow$

" $\Rightarrow$ " Contrad: 1 & 2 hold.

$\exists\, x^*: Ax^* = 0$, $x^* > 0 \Rightarrow \forall y, yAx^* = 0$

$\Updownarrow$

$\exists\, y^*: y^*A \gg 0 \Rightarrow \forall x > 0, y^*Ax > 0$

Contradiction!

"$\Leftarrow$" $\quad yA \gg 0 \iff \exists s > 0 : yA \geq s \cdot e$.

$e = (1, \ldots, 1)$.

s = smallest element in yA.

$yA \gg 0$ has no solution iff:

$$(y, -s) \begin{bmatrix} A \\ e \end{bmatrix} \geq 0 \; ; \; (0, \ldots, 0, 1) \begin{bmatrix} y \\ -s \end{bmatrix} < 0$$

$\Longrightarrow$ By Farka's lemma:

$$\begin{bmatrix} A \\ e \end{bmatrix} x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{Ax = 0}$$

$$ex = 1$$

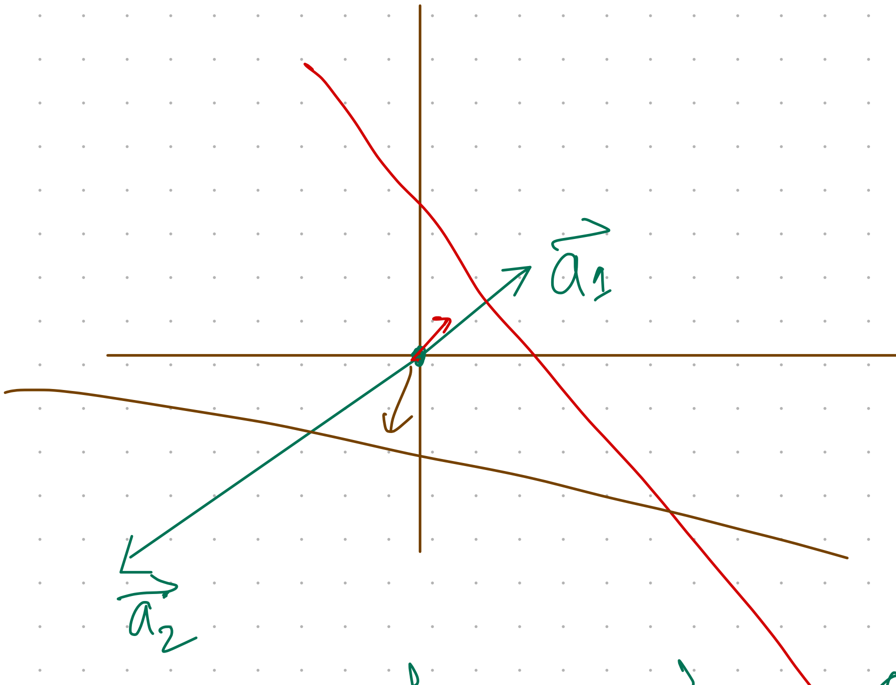
$yA \gg 0 \iff yA - se \geq 0$.

$$(y, -s) \begin{bmatrix} A \\ e \end{bmatrix} \; ; \; \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} (y, -s).$$

more negative than 0.

$-s < 0$.

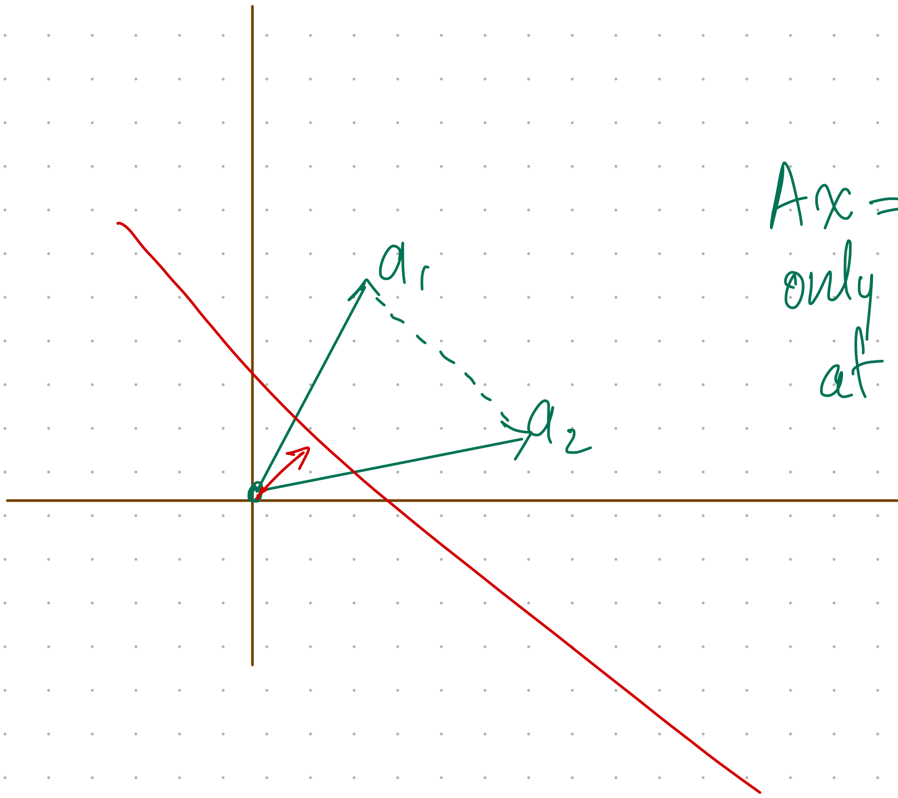
$\boxed{\text{Eg } 1}$ $A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$.



$x_1 \vec{a}_1 + x_2 \vec{a}_2 = 0$, $x > 0$.

$(x_1, x_2) = (2, 1)$.

Eg 2 : $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$



$Ax = 0$
only possible
at $x = 0$.