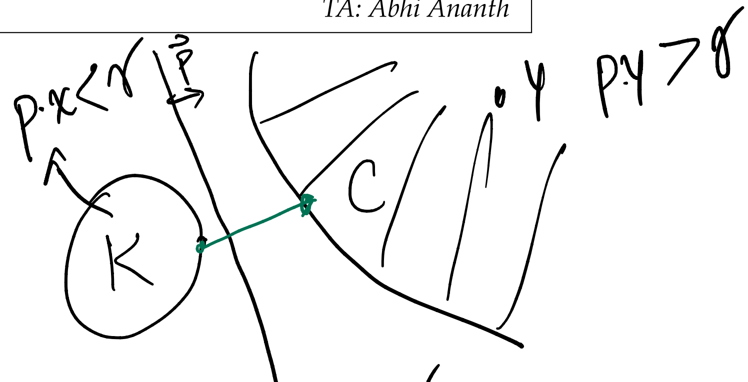
ECON 6100		03/05/2021
	Section 3	
Lecturer: Larry Blume		*TA: Abhi Ananth*



* These notes develop Fikri Pitsuwan's notes from 2017.

1 Review

1.1 Separating Hyperplane Theorem

Theorem. *Let K and C be disjoint nonempty convex subsets of $\mathbb{R}^n$. Suppose K is compact and C is closed. Then there exists a nonzero $p \in \mathbb{R}^n$ that strongly separates K and C. That is,*

$$\underline{\forall x \in K, p \cdot x < \gamma \quad \text{and} \quad \forall y \in C, p \cdot y > \gamma}$$

for some $\gamma \in \mathbb{R}$.

Now, let's use this to show one direction of Farka's Lemma. For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, if there exists no $x \geq 0$ such that $Ax = b$, then there must exist a $y \in \mathbb{R}^n$ such that $yA \geq 0$ and $y \cdot b < 0$.

$a_i = i^{th}$ Coln of A. $A = [a_1, a_2, \ldots, a_n]$

$C = \{A \cdot x : x \geq 0\}$.

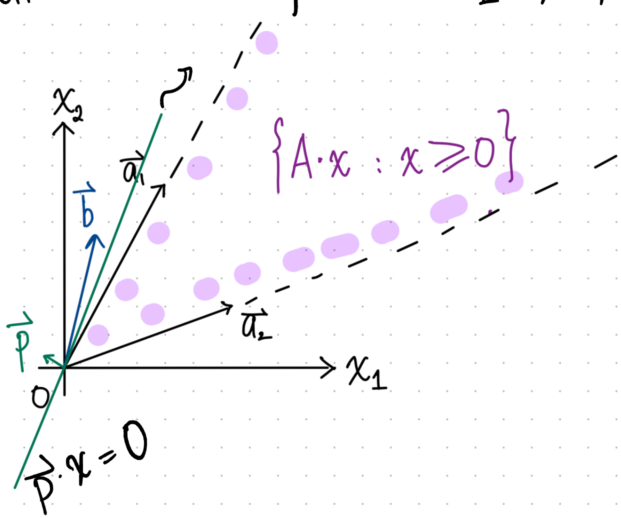
$\downarrow$

closed convex

$K \longrightarrow$ compact, convex

$\|$

$\{\vec{b}\}$



Applying SHT: $\exists \vec{p}, \gamma$: $\underbrace{p \cdot \vec{b} > \gamma}_{B}$ but $\forall s \in C$: $\vec{p} \cdot \vec{s} < \gamma$

$\lambda \cdot \vec{a}_1 \in C, \quad \forall \lambda \geq 0 \Rightarrow \vec{p} \cdot (\lambda \vec{a}_1) < \gamma$

$\vec{p} \cdot \vec{a}_1 < \gamma / \lambda$.

Take limit as $\lambda \to \infty$.

$\vec{p} \cdot \vec{a}_1 \leq 0$

$\lambda \cdot \vec{a}_2 \in C \;, \forall \lambda \geq 0 \Rightarrow \vec{p} \cdot (\lambda \vec{a}_2) < \gamma$.

$\left\{ \begin{array}{l} p \cdot a_2 < \gamma/\lambda \\ \text{As } \lambda \to \infty \Rightarrow p \cdot a_2 \leq 0 \end{array} \right.$

$\vec{p} \left[\underbrace{\vec{a}_1 , \vec{a}_2} \right] \leq [0, 0] \Leftrightarrow \vec{p} \cdot A \leq \vec{0}$

$p[\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_n] \leq [0, \cdots, 0]$

Clarifying.

$p \cdot a_2 < \gamma/\lambda \Rightarrow p \cdot a_2 \leq \gamma/\lambda$. $\forall \gamma \in \mathbb{R}$.

$\Rightarrow \lim_{\lambda \to \infty} p \, a_2 \leq \underbrace{\lim_{\lambda \to \infty} \gamma/\lambda}_{=0}$.

$\left. \begin{array}{l} a \leq b. \\ -a \geq -b. \end{array} \right\}$

$\vec{p} A \leq 0 \Leftrightarrow \underbrace{(-\vec{p})}_{=\vec{y}} A \geq 0 \qquad \vec{y} A \geq 0$.

From (B) $\quad \vec{p} \cdot \vec{b} > \gamma$

$-\vec{y} \cdot \vec{b} > \gamma \Rightarrow \vec{y} \cdot \vec{b} < -\gamma \leq 0$.

$C \ni \{\vec{0}\}$. $\quad \left. \begin{array}{l} \vec{p} \cdot \vec{0} = 0. \\ \vec{p} \cdot \vec{b} > \gamma. \\ \vec{p} \cdot c < \gamma \end{array} \right\} \Rightarrow \gamma > 0$.

$\forall c \in C$

By contradiction.

If $\gamma = 0 \rightarrow \vec{p} \cdot c < \gamma$.

$0 < 0 \cdot \times$.

1.2 Complementary Slackness

Given a linear program (P) in canonical form

$$\begin{aligned} v_p(b) &= \max c \cdot x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

The *dual* linear program (D) is

$$\begin{aligned} v_D(c) &= \min y \cdot b \\ \text{s.t. } & yA \geq c \\ & y \geq 0 \end{aligned}$$

A very useful theorem from the study of duality is the complementary slackness theorem.

Theorem. *If x^* and y^* are feasible for the primal and dual, then they are optimal if and only if $y^*(b - Ax^*) = 0$ and $(y^*A - c)x^* = 0$*

Consider the situation where both problems are feasible. That is the both:

$$\{y \geq 0 : yA \geq c\} \neq \emptyset \text{ and } \{x \geq 0 : Ax \leq b\} \neq \emptyset$$

For any $x \geq 0, Ax \leq b$ and any $y \geq 0, yA \geq c$, then

$Ax \leq b. \overset{y \geq 0}{\Rightarrow} yAx \leq yb.$

$c \leq yA \underset{x \geq 0}{\Rightarrow} cx \leq yAx.$

$$c \cdot x \leq yAx \leq y \cdot b \qquad \text{(Why?)}$$

Suppose x^* solves P and y^* solves D, then by the duality theorem $\boxed{c \cdot x^* = y^* \cdot b}$. So,

$$c \cdot x^* = y^*Ax^* = y^* \cdot b$$

1.3 Leontief Production Models

A *simple Leontief model* is a model of the production sector. There are $n + 1$ factors of production consisting of 1 primary factor (think of this as labor) and n produced factors. We ignore labor and first focus on the interdependency of the produced factors.

The main ingredient of the model is the $n \times n$ *input requirement* or *activity* matrix A, where the elements are non-negative, $a_{ij} \geq 0$, and none of the rows are all zeros, for all i, there is $a_{ij} > 0$.

$a_{i \to j}$: using i to make j

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Recipe to produce 1 unit of good 1.

a_{ij} = amount of good i needed to produce 1 unit of good j. The column $A^j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$ then describes the amount of all goods required to produce 1 unit of good j.

Reqd. out $= (1, 1)$.

Reqd. input $= \begin{pmatrix} 1/2 + 1/4 \\ 1/4 + 1/2 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 3/4 \end{pmatrix}$

Example 1 (Input requirement matrix A).

$$A = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$$

What property do we want to impose on A? If you think of A as characterizing a production process or a machine, a natural property is to demand that what you get out of the process is more than what you put in. Formally, let $x \in \mathbb{R}^n$ be a vector of gross output (think of this as what the machine spits out) then the machine requires Ax amount of inputs. What you want is the output to be more than the input.

Definition 1. The matrix A is *productive* if there is $x^* \geq 0$ such that $x^* \gg Ax^*$.

This is saying that your machine described by A is not useless: it can produce some stuff, using less stuff.

Example 2 (Productive matrix A).

$$A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

$x \gg Ax$.

Example 3 (Non-productive matrix A).

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow A \cdot x = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$x_1 > x_2$

$x_2 > x_1$

Theorem. *The matrix A is productive if and only if for all $y \geq 0$, $(I - A)x = y$ has a* non-negative *solution, i.e., there is an $x \geq 0$ such that $(I - A)x = y$.*

$(I-A)x = x - Ax \rightarrow$ net outpup.

The theorem says that if the matrix A can produce something, it can produce anything! This is not too surprising when you think about it since we have linear production. Roughly speaking, if A is productive, so it can produce some stuff. If you want to scale the output, then you can just scale the input appropriately linearity.

Now, the above analysis imposes no constraint on production, so when A is productive you can produce any amount you want. Indeed, the missing ingredient we need for the model to be useful is labor, the primary factor of production.

The labor requirement vector is given by $a_0 = (a_{01}, \ldots a_{0n})$, where a_{0j} is the amount of labor needed to produce one unit of output j. Let L be the supply of labor. It now makes sense to describe what we can produce.

Definition 2. The set of *feasible net output* is

Total Supply Labor

$$Y = \{y \in \mathbb{R}^n : a_0 \cdot (I - A)^{-1} y \leq L, y \geq 0\}$$

$\hookrightarrow a_0 = (a_{01}, \ldots, a_{0n})$: a_{0i} = amount of lab. reqd. for 1 unit of good i.

We have a constraint on how much we can produce. A natural question then would be how much should we produce? To do this we need to think about costs and revenue, so we need prices. Let $p = (p_1, \ldots, p_n)$ be the prices of goods and let w denote the wage rate. The profit from 1 unit of good i would be

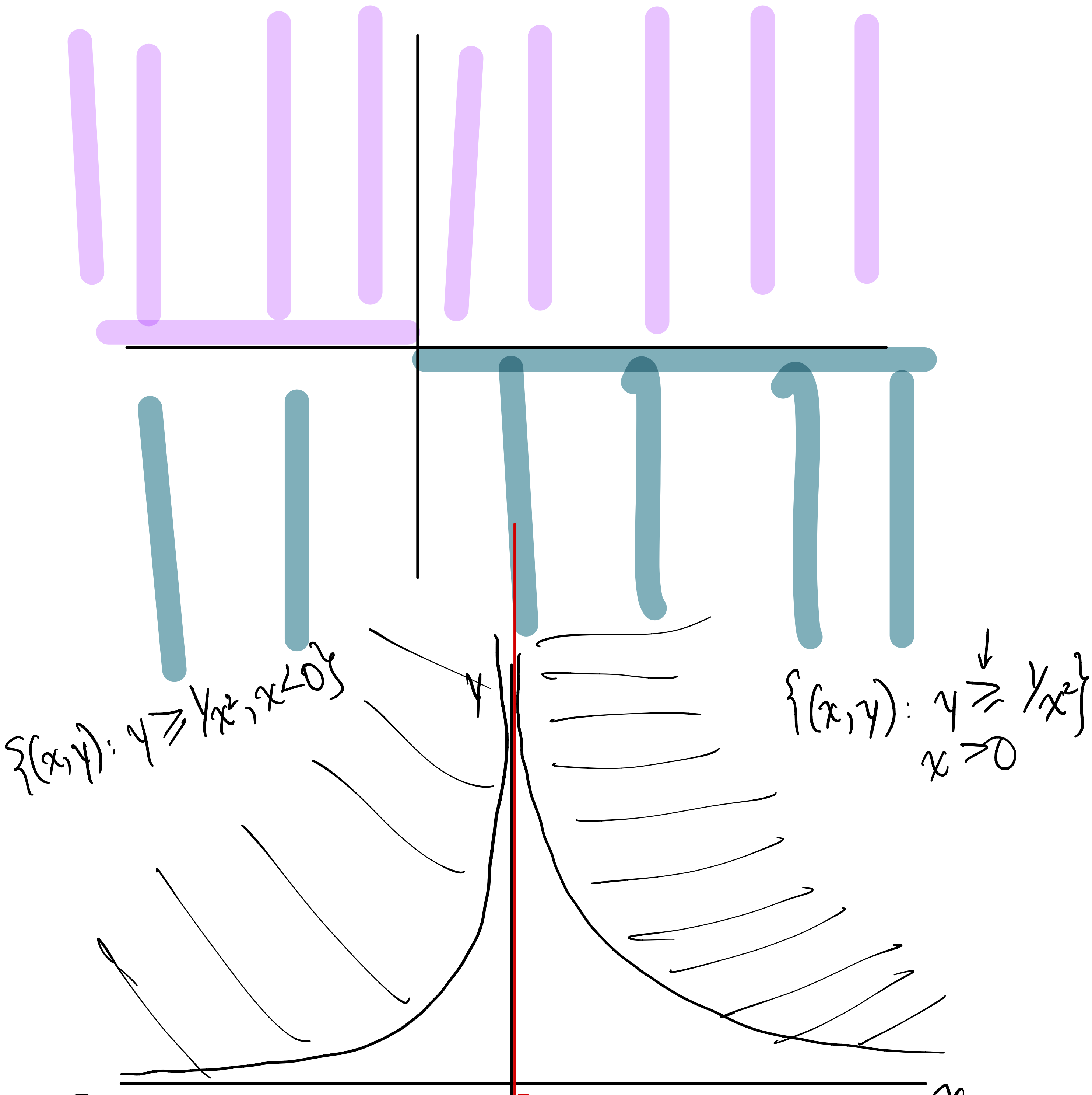
$$\pi_i = p_i - (wa_{0i} + p_1 a_{1i} + \cdots + p_n a_{ni})$$

Rearranging and putting this in vector form gives the rate of profit $\pi = p(I - A) - wa_0$. With gross output vector x, the profit is $\pi \cdot x$.

Theorem 1. *If A is productive and $a_0 \gg 0$, then there is a $(w^*, p^*) \gg 0$ such that $\pi^* = 0$. This is given by $p^* = w^* a_0 (I - A)^{-1}$ for any $w^* > 0$.*

2 Problems

Problem 1. Example of two *disjoint* closed convex sets that cannot be strongly separated.



A $\quad \vec{p}\cdot\begin{pmatrix}x\\y\end{pmatrix} \geq \gamma$

$\{(x,y): x=0\}$ $\quad \vec{p}\begin{pmatrix}x\\y\end{pmatrix} < \gamma. \quad$ B

$p=[1,0]$

Q: (By Yiqi): Why doesn't $p = (-1, 0)$, $\gamma = 0$ work?

<u>Ans:</u> $\forall (x,y) \in \{(x,y) : y \geq 1/x^2 , x < 0\}$

$$\vec{p} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -x = |x| > 0.$$

$\forall (x,y) \in \{(x,y) : y \geq 1/x^2 , x > 0\}$

$$\vec{p} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -x = -|x| < 0.$$

↳ ⊛ This is true. Good catch Yiqi!

Turns out this is due to the definition of strict separation we use.

The example would work for the following definition:

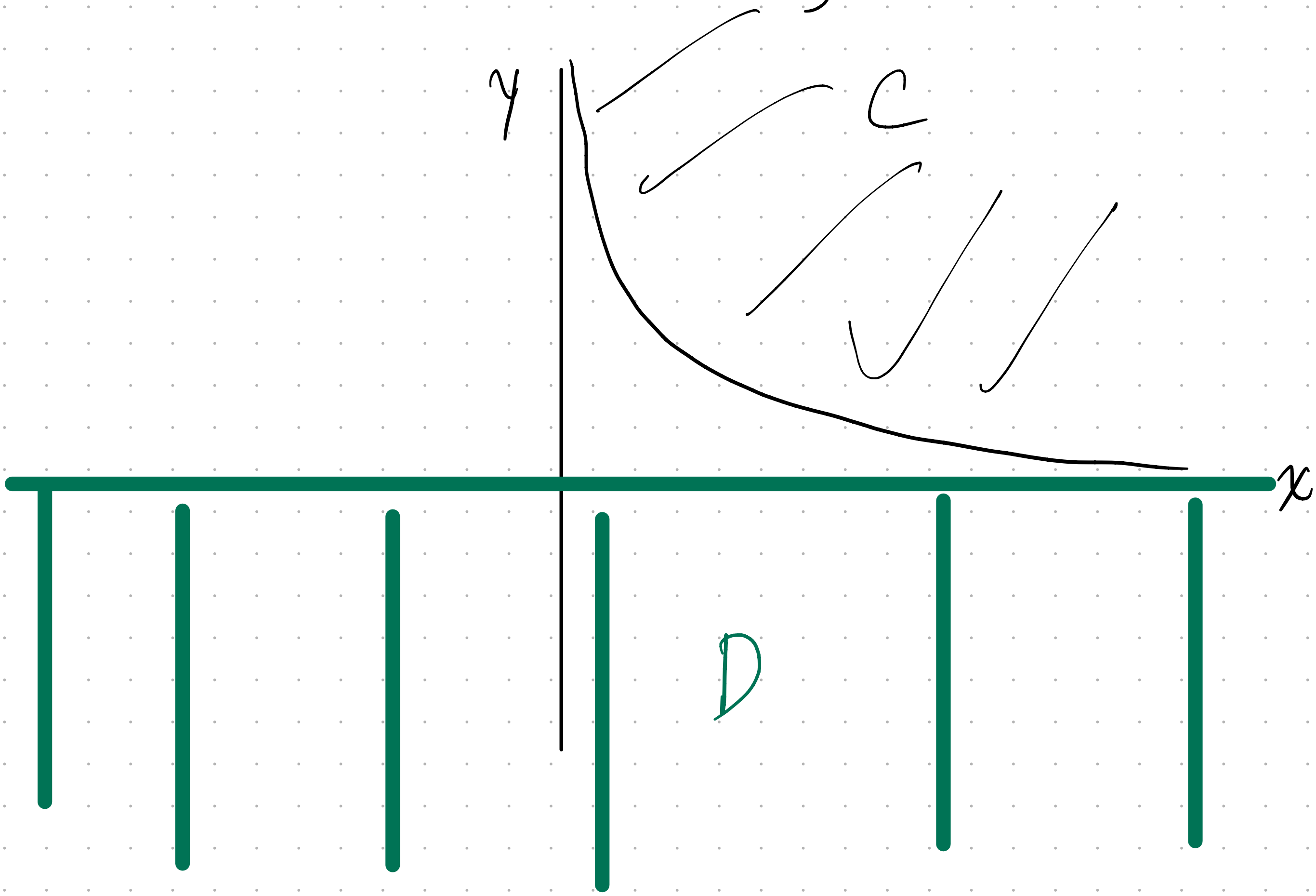
"Two convex sets $C, D \subseteq \mathbb{R}^n$ are strictly separated if $\exists \vec{p} : \sup_{\vec{x} \in C} \vec{p} \cdot \vec{x} < \inf_{\vec{x} \in D} \vec{p} \cdot \vec{x}$."

⊛ As it turns out, these definitions are not equivalent.

An example that works:

Suppose $C = \{(x,y) : x > 0,\ y \geq 1/x\}$

$D = \{(x,y) : y \leq 0\}$



Now, there is no strict separation.

Suppose $\vec{p} = [0,1]$, $\gamma = 0$:

$$\forall (x,y) \in C : \vec{p} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = y > 0$$

$$\forall (x,y) \in D : \vec{p} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = y$$

Since $(1,0) \in D$, $\vec{p} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \not< 0$.

Problem 2. You are playing *Settler's of Simpler Catan*. You are endowed with 4 units of resource *wood* and 8 units of resource *sheep*. In order to win, you must collect the maximum number of victory points. Each *settlement* is worth 1 victory points and each *development card* is worth 2 victory points. You must decide how many *settlements* you want to build and how many *development cards* to accumulate.

Each *settlement* costs 1 unit of *wood* and 1 unit of *sheep*. Each *development card* costs 1 unit of *wood* and 3 units of *sheep*.

Jeff Bezos enters the game and wants to buy your resources. What is the minimum amount he must pay you (in victory points) so that you will be no worse off? What is price offered per resource?

$$\max \quad 1 \cdot x_S + 2 x_D \quad = V_P$$

$$\text{s.t.} \quad 1 x_S + 1 x_D \leq 4 \quad (y_w)$$

$$1 x_S + 3 x_D \leq 8 \quad (y_s)$$

$$x_S, x_D \geq 0$$

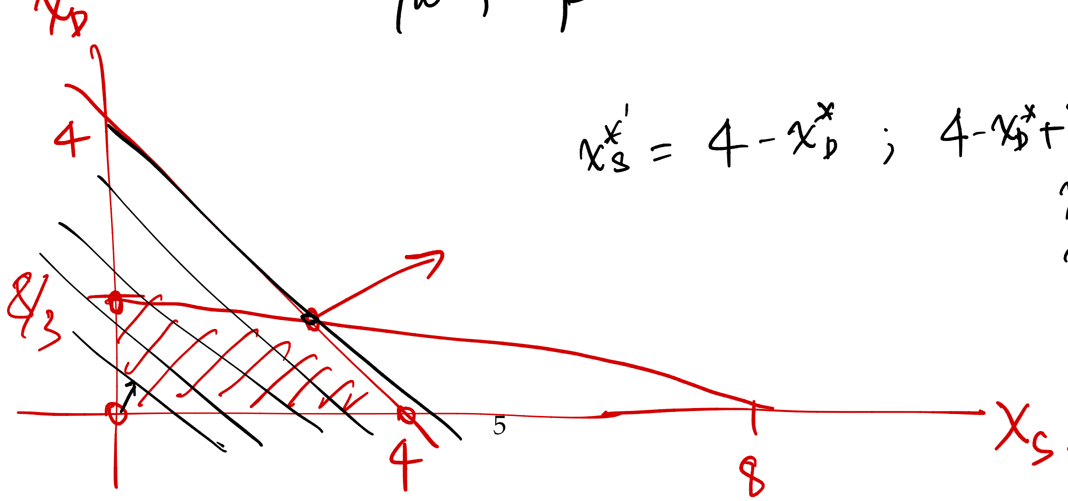
$$Ax \leq b \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \qquad yA \geq c.$$

$$\min \quad 4 y_w + 8 y_s$$

$$y_w + y_s \geq 1$$

$$y_w + 3 y_s \geq 2$$

$$y_w, y_s \geq 0.$$



$x_S^{*\prime} = 4 - x_D^* \; ; \; 4 - x_D^* + 3x_D^* = 8.$

$x_D^* = 2.$

$x_S^* = 2.$

$$\left.\begin{aligned} y^*(b - Ax^*) &= 0 \\ (c - y^*A)x^* &= 0 \end{aligned}\right\} \longrightarrow y^* = (1/2, 1/2)$$

Problem 3. Suppose the activity matrix for a simple Leontief model is $A = \begin{bmatrix} 1/3 & 1/4 \\ a & 1/2 \end{bmatrix}$ and labor input requirements are $a_0 = (1/3, 1/2)$.

(a) For what values of a is A productive?

(b) Let $L = 5$, describe the set of feasible net outputs if $a = 1$.

~~(c) Let $w = 1$, describe the set of equilibrium prices in terms of a.~~

(a) A is productive iff $(I-A)^{-1}$ is non-neg.

$B = (I-A)^{-1}$

$By \geq 0 \quad \forall y \geq 0.$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; \quad A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(I-A) = \begin{bmatrix} 1-1/3 & -1/4 \\ -a & 1-1/2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/4 \\ -a & 1/2 \end{bmatrix}.$$

$$(I-A)^{-1} = \underbrace{\frac{1}{1/3 - a/4}}_{\geq 0} \begin{bmatrix} 1/2 & 1/4 \\ \underbrace{a}_{\geq 0} & 2/3 \end{bmatrix}$$

$0 \leq a \leq 4/3$