ECON 6100 03/12/2021

Section 4

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1 Review

A *simple Leontief model* is a model of the production sector. There are $n+1$ factors of production consisting of 1 primary factor (think of this as labor) and n produced factors. We ignore labor and first focus on the interdependency of the produced factors.

The main ingredient of the model is the $n \times n$ *input requirement* or *activity* matrix A, where the elements are non-negative, $a_{ij} \geq 0$, and none of the rows are all zeros, for all i, there is $a_{ij} > 0$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

a_{ij} = amount of good i needed to produce 1 unit of good j. The column $A^j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$ then describes the amount of all goods required to produce 1 unit of good j.

Definition 1. The matrix A is *productive* if there is $x^* \geq 0$ such that $x^* \gg Ax^*$.

Theorem 1. *The matrix A is productive if and only if for all $y \geq 0$, $(I-A)x = y$ has a non-negative solution, i.e., there is an $x \geq 0$ such that $(I-A)x = y$.*

Now, the above analysis imposes no constraint on production, so when A is productive you can produce any amount you want. Indeed, the missing ingredient we need for the model to be useful is labor, the primary factor of production.

I've gathered here the interesting features of a productive matrix A. Some of these will be useful for your problem set:

1. If A is productive and $x \geq Ax$, then $x \geq 0$.
2. If A is productive, then $(I-A)$ has full rank.
3. If A is productive, then $A^n x \to 0$ and $n \to 0$.

The labor requirement vector is given by $a_0 = (a_{01}, \dots a_{0n})$, where a_{0j} is the amount of labor needed to produce one unit of output j. Let L be the supply of labor. It now makes sense to describe what we can produce.

Definition 2. The set of *feasible net output* is

$$Y = \{y \in \mathbb{R}^n : a_0 \cdot (I-A)^{-1} y \leq L, y \geq 0\}$$

We have a constraint on how much we can produce. A natural question then would be how much should we produce? To do this we need to think about costs and revenue, so we need prices. Let $p = (p_1, \ldots, p_n)$ be the prices of goods and let w denote the wage rate. The profit from 1 unit of good i would be

$$\pi_i = p_i - (wa_{0i} + p_1 a_{1i} + \cdots + p_n a_{ni})$$

Rearranging and putting this in vector form gives the rate of profit $\pi = p(I - A) - wa_0$. With gross output vector x, the profit is $\pi \cdot x$.

Theorem 2. *If A is productive and $a_0 \gg 0$, then there is a $(w^*, p^*) \gg 0$ such that $\pi^* = 0$. This is given by $p^* = w^* a_0 (I - A)^{-1}$ for any $w^* > 0$.*

2 Problems

Problem 1. Suppose the activity matrix for a simple Leontief model is $A = \begin{bmatrix} 1/3 & 1/4 \\ a & 1/2 \end{bmatrix}$ and labor input requirements are $a_0 = (1/3, 1/2)$.

(a) For what values of a is A productive?

(b) Let $L = 5$, describe the set of feasible net outputs when $a = 1$.

(c) Let $w = 1$, describe the set of equilibrium prices in terms of a. → ★ Assume $0 \le a < 4/3$.

a) A is productive iff $(I-A)^{-1}$ exists & non-negative.

$$I - A = \begin{bmatrix} 2/3 & -1/4 \\ -a & 1/2 \end{bmatrix}$$

$$(I-A)^{-1} = \frac{1}{\underbrace{(1/3 - a/4)}_{>0}} \begin{bmatrix} 1/2 & 1/4 \\ \underbrace{a}_{\geq 0} & 2/3 \end{bmatrix}$$

when $a = 1$

$$\frac{1}{1/12}\begin{bmatrix} 1/2 & 1/4 \\ 1 & 2/3 \end{bmatrix}$$

$$\boxed{0 \le a < 4/3}$$

b) $Y = \{ y \in \mathbb{R}^2 : \vec{a}_0 \underbrace{(I-A)^{-1} y}_{x} \le L, \; y \geq 0 \}$.

$a_0 = (1/3, 1/2)$; $(I-A)^{-1} = \begin{bmatrix} 6 & 3 \\ 12 & 8 \end{bmatrix}$

$a_0 (I-A)^{-1} = (8 \;\; 5)$

$Y = \{ y \in \mathbb{R}^2 : 8y_1 + 5y_2 \le \overset{L}{5}, \; y \geq 0 \}$.

c) A is productive:

$$p^* = \cancel{w^*} a_0 [I-A]^{-1}$$

$$= \begin{pmatrix} 1/3 & 1/2 \end{pmatrix} \begin{bmatrix} \frac{1}{2} \frac{12}{4-3a} & \frac{1}{4} \frac{12}{4-3a} \\ a \frac{12}{4-3a} & \frac{2}{3} \frac{12}{4-3a} \end{bmatrix}$$

$$= \left(\underbrace{\frac{2+6a}{4-3a}}_{a < 4/3} , \frac{5}{4-3a} \right) \gg (0,0).$$

Problem 2. Consider the production matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and its associated digraph

$$G = \begin{bmatrix} 1\{a_{11} > 0\} & 1\{a_{12} > 0\} & 1\{a_{13} > 0\} \\ 1\{a_{21} > 0\} & 1\{a_{22} > 0\} & 1\{a_{23} > 0\} \\ 1\{a_{31} > 0\} & 1\{a_{32} > 0\} & 1\{a_{33} > 0\} \end{bmatrix}$$

How does **irreducibility** of A related to the **strong connectedness** of G?

$\underset{n\times n}{A}$ is reducible if $\{1, \dots, n\}$ can be partitioned into S, S^c such that .

permutation $\leftarrow \{l_1, \dots, l_n\}$ of $\{1, \dots, n\}$

$S = \{l_1, \dots, l_p\}$

$S^c = \{l_{p+1}, \dots, l_n\}$

$$\tilde{A} = \begin{array}{cc} & \begin{array}{cc} \overbrace{l_1 \; \cdots \; l_p}^{S} & \overbrace{l_{p+1} \; \cdots \; l_n}^{S^c} \end{array} \\ \begin{array}{c} S\left\{\begin{array}{c} l_1 \\ l_2 \\ \vdots \\ l_p \end{array}\right. \\ S^c\left\{\begin{array}{c} l_{p+1} \\ \vdots \\ l_n \end{array}\right. \end{array} & \left[\begin{array}{c|c} A_{SS} & A_{SS^c} \\ \hline A_{S^cS} & A_{S^cS^c} \end{array}\right] \end{array}$$

$A_{S^cS} = 0$ if A is reducible.

If I have goods in S^c, I can only use it to produce goods S^c.

G is strongly connected:

$\forall i, j$: i - exists a path from i to j $(i \to j)$

ii " " " j to i $(j \to i)$

Define:

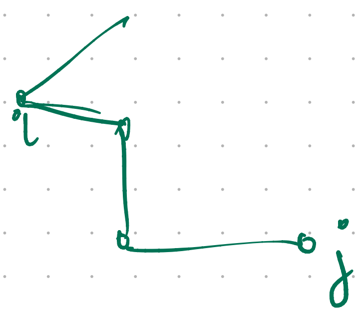
$\forall i$: $O(i) \equiv$ set of nodes that you can reach starting from i

"outward paths from i"

$I(i) \equiv$ set of nodes you can reach node i from

"inward paths into i".

G is str. conn : $\iff$ $\forall i,j$, $\begin{array}{c} j \in O(i) \\ \Updownarrow \\ i \in I(j) \end{array}$, $\begin{array}{c} i \in O(j) \\ \Updownarrow \\ j \in I(i) \end{array}$.



Stat: A is irreducible $\iff$ G is str. conn.

$g_{ij} = \mathbb{1}\{a_{ij} > 0\}$

G is str conn $\Rightarrow$ A is irred. $\longleftrightarrow$ A is reducible $\Rightarrow$ G not str. conn.

A is irred. $\Rightarrow$ G is str. conn. $\longleftrightarrow$ G is not str. conn $\Rightarrow$ A is red.

A is reducible $\Longrightarrow \exists\ S, S^c$ partition :

$$A = \begin{array}{c c} & \begin{array}{cc} S \ (\to l_i) & S^c \end{array} \\ \begin{array}{c} S \\ l_j \leftarrow S^c \end{array} & \left[\begin{array}{c|c} A_{SS} & A_{SS^c} \\ \hline 0 & A_{S^cS^c} \end{array}\right] \begin{array}{c} l_1 \\ \vdots \\ l_n \end{array} \end{array}$$

$g_{l_i l_j} = 0$. $\quad (\because a_{l_i l_j} = 0)$

It is implies $\quad l_i \notin O(l_j)$

$i \notin O(j) \Longrightarrow G$ is not str. conn.

G is not str. conn $\Longrightarrow$ A is reducible.

$\downarrow$

$\exists\ i, j$ such $\quad i \notin O(j) \iff j \in I(i)$ $\quad j \not\to i$

<u>Claim:</u> $O(j) \cap I(i) = \phi$

$O(j)$: nodes reached using paths out from j

$I(i)$: paths in to i — "nodes reached using"

Towards a contr. : suppose $k \in O(j) \cap I(i)$.

$\Longrightarrow j \longrightarrow k \ ; \ k \longrightarrow i$

$\Longrightarrow j \longrightarrow i$ ✗ Contradiction.

$S^c = O(j)$

$$\begin{array}{c c} & \begin{array}{cc} S & S^c \end{array} \\ \begin{array}{c} S \\ S^c \end{array} & \left[\begin{array}{c|c} - & - \\ \hline 0 & \end{array}\right] \end{array} \doteq A.$$

G

WHY? $\forall i \in S^c, j \notin S^c$ $g_{ij} = 0 \Rightarrow a_{ij} = 0$

Problem 3. Suppose you have an activity matrix as given below:

$$A = \begin{bmatrix} 0.1 & 0.15 & 0.12 \\ 0.2 & 0 & 0.3 \\ 0.25 & 0.4 & 0.2 \end{bmatrix}$$

You'd like a net output of

$$y = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

What's gross out required to generate this net output?

$y = x - Ax = y = (I-A)x.$

To calculate the inverse of the matrix: `https://matrix.reshish.com/inverse.php`
Discuss how to interpret $(I - A)^{-1}$.

exists & non-neg. $\Longleftrightarrow$ A is prod.

Given y: $x = (I-A)^{-1} y \leftarrow$

$$= \begin{bmatrix} 281.30 \\ 464.86 \\ 695.34 \end{bmatrix}$$

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1-a_{33} \end{bmatrix}^{-1} = (b_{ij})_{1 \le i,j \le n}.$$

HINT: $e^1 = (1, 0, \dots, 0)$.

$\tilde{y} = y + e^1 \longrightarrow \bar{x} = ?$

$$\left.\begin{aligned} \bar{x} &= (I-A)^{-1}(y+e^1) \\ x &= (I-A)^{-1} y \end{aligned}\right\} \longrightarrow \bar{x} - x = \underbrace{(I-A)^{-1} e^1}$$

(*) If the i^{th} component of y changes by 1, the i^{th} column of $(I-A)^{-1}$ informs us how each good must adjust to generate this extra good.

Problem 4. Suppose you have an activity matrix as given below:

$$A = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.3 & 0.1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1, 2 dont require 3.
3 doesn't require 1, 2.

You'd like a net output of

$$y = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

What's gross out required to generate this net output? What if $a_{33} = 0.99$?

$$(I-A) = \begin{bmatrix} 0.8 & -0.1 & 0 \\ -0.3 & 0.9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{singular.}$$

$(I-A)^{-1} \rightarrow$ doesn't exist.

$$(I-A)x = y.$$

With block diagonal A,

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

Treat A_1 & A_2 as if they are independent economic.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (I-A_1)^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}$$

$y_1 \rightarrow 0$, $y_2 \rightarrow 0$

A_1's prod.

A_1 is productive

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$I - A_1$ is full rank.

$$(I-A_1)x = \vec{0}.$$

A_2 ? A_2 [1].

$$(I_1 - A_2)\, x_3 = y_3 \longleftrightarrow 0.$$

$$x_3 \geq 0$$