ECON 6100	03/19/2021
Section 5	
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Developed from Fikri Pitsuwan's notes.

1 Review

The *General Leontief Model (GLM)* extends the Simple Leontief Model (SLM) to capture the notion of multiple productive technology.

The main ingredients of the model are the $n \times m$ *input* matrix A and the $n \times m$ *output* matrix B, all with non-negative entries.

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & \cdots & b_{1m} \\ \vdots & b_{22} & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ b_{n1} & \dots & \dots & \dots & b_{nm} \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1m} \\ \vdots & a_{22} & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ a_{n1} & \dots & \dots & \dots & a_{nm} \end{bmatrix}$$

Definition 1. (A, B) is productive if there exists an $x \geq 0$ such that $(B - A) \cdot x >> 0$.

Assumptions maintained.

1. (A, B) is productive
2. No joint production
3. Every good is produced

Definition 2. A technology t is a set of n activities such that each good is produced by only one activity.

Lemma 1. *If a GLM (A, B) is productive and produces $y \gg 0$, then there is a technology that produces y.*

Consider the example:

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \qquad B - A = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Take $y' = (1/2, 3/2)^T$, verify that both technologies $t = \{1, 2\}$ and $s = \{2, 3\}$ can produce y since

$$\begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

A natural question now is to ask which technology should we use? To compare technologies, we need some notion of efficiency of a technology. To that end, we introduce a finite resource, labor. For the GLM, the labor requirement vector is given by $a_0 = (a_{01}, \ldots a_{0m})$, where a_{0j} is the amount of labor needed to run one unit of activity j. Let L be the supply of labor. The set of *feasible net output (PPS)* for the GLM is $Y = \{y \in \mathbb{R}^n, y \geq 0 : y \leq (B-A)x, a_0 \cdot x \leq L, x \geq 0\}$.

The best way to produce y' is now the most efficient way to produce y'. Here, that will correspond to using the least amount of the finite resource, labor.

Example 1. Let $a_0 = (1, 1/2, 1/2)$, then to produce y', technology t needs $a_0 \cdot (3,4,0)^T = 5$, while s needs $a_0 \cdot (0,1,3)^T = 2$. Thus, we should use technology s to produce y'.

Our analysis shows that some technologies are better than others, but what if we allow all activities to be used. Can we do any better? The answer is no.

For a given y, the following linear program gives the activity vector x that minimizes labor used.

$$\begin{aligned} \lambda(y) &= \min a_0 \cdot x \\ \text{s. t. } (B-A)x &\geq y \\ x &\geq 0 \end{aligned} \qquad (\mathcal{L}(y))$$

For a given y and a technology t, the following linear program gives the activity vector using technology t that minimizes labor used.

$$\begin{aligned} \lambda^t(y) &= \min a_0 \cdot x \\ \text{s. t. } (B-A)x &\geq y \\ x &\geq 0 \\ x_i &= 0 \text{ if } i \notin t \end{aligned} \qquad (\mathcal{L}^t(y))$$

First note that for a given y', if x is an optimal solution to $\mathcal{L}$, then there is a basic optimal solution which is an optimal solution to $\mathcal{L}^t$ for some t. That is, $\lambda(y') = \lambda^t(y')$ for some t. With the same reasoning, for a given y'', there is a technology s such that $\lambda(y'') = \lambda^s(y'')$. There is no reason for s and t to be the same in our reasoning, but the next theorem says that they are.

Theorem 2 (Non-substitution)**.** *If a GLM (A, B) satisfies the maintained assumptions, there is a technology t such that for all $y \geq 0$, $\lambda^t(y) = \lambda(y)$.*

Example 2. Let $y' = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$ and $y'' = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$. The problems are

$$\begin{aligned} \lambda(y') &= \min x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \text{s. t. } -x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 3 \\ x &\geq 0 \end{aligned} \qquad (\mathcal{L}(y'))$$

$$\begin{aligned} \lambda(y'') &= \min x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \text{s. t. } -x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 4 \\ x &\geq 0 \end{aligned} \qquad (\mathcal{L}(y''))$$

The solutions are $x' = (0,1,3)^T$ and $x'' = (0,1,4)^T$, respectively. We see that technology $s = \{2,3\}$ gives $\lambda^s(y') = \lambda(y')$ and also that $\lambda^s(y'') = \lambda(y'')$.

The non-substitution theorem says that technologies can be ranked by how efficient they are, where efficiency of a technology is measured by how much labor is used. Another interpretation of the non-substitution theorem is that when a GLM is productive, has one primary factor, and satisfies all our assumptions, restricting our usage to technology t with n activities does not alter the production possibility.

2 Problems

Problem 1 (Q7 of Linear Economic Models)**.**

Problem 2. Consider a GLM given by

$$B = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1/2 & 1/4 \end{bmatrix}$$

and suppose that there are two primary factors $a_P = (1, 0)$ and $a_E = (0, 1/2)$ with labor supply $L_P = 4$ and $L_E = 2$, respectively. Is there a technology that does not alter the production possibility? Compare this to the case with one primary factor $a_0 = (1, 1/2)$ and $L = 6$.