ECON 6100	03/19/2021

Section 5

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Developed from Fikri Pitsuwan's notes.

1 Review

The *General Leontief Model (GLM)* extends the Simple Leontief Model (SLM) to capture the notion of multiple productive technology.

The main ingredients of the model are the $n \times m$ *input* matrix A and the $n \times m$ *output* matrix B, all with non-negative entries.

Activity (m)

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & \cdots & b_{1m} \\ \vdots & b_{22} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ b_{n1} & \cdots & \cdots & \cdots & b_{nm} \end{bmatrix} \text{goods (n)} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1m} \\ \vdots & a_{22} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nm} \end{bmatrix}$$

Definition 1. (A, B) is productive if there exists an $x \geq 0$ such that $(B - A) \cdot x >> 0$.

Assumptions maintained. $n < m$ $\quad |Z| = n \subseteq \{1..,m\}$

1. (A, B) is productive
2. No joint production $\rightarrow$ Any column/activity in B has at most 1 element > 0
3. Every good is produced $\rightarrow$ Any row/good in B has at least 1 element > 0

Definition 2. A technology t is a set of n activities such that each good is produced by only one activity.

Lemma 1. *If a GLM (A, B) is productive and produces $y \gg 0$, then there is a technology that produces y.*

Consider the example:

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \qquad B - A = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

t s

Take $y' = (1/2, 3/2)^T$, verify that both technologies $t = \{1, 2\}$ and $s = \{2, 3\}$ can produce y since

$$\begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

A natural question now is to ask which technology should we use? To compare technologies, we need some notion of efficiency of a technology. To that end, we introduce a finite resource, labor. For the GLM, the labor requirement vector is given by $a_0 = (a_{01}, \ldots a_{0m})$, where a_{0j} is the amount of labor needed to run

$a_0 \cdot \vec{x} \rightarrow$ scalar

$\underline{a_0} \cdot \vec{x} \rightarrow$ vector.

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ vs $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

one unit of activity j. Let L be the supply of labor. The set of *feasible net output (PPS)* for the GLM is $Y = \{y \in \mathbb{R}^n, y \geq 0 : y \leq (B - A)x, a_0 \cdot x \leq L, x \geq 0\}$.

The best way to produce y' is now the most efficient way to produce y'. Here, that will correspond to using the least amount of the finite resource, labor.

Example 1. Let $a_0 = (1, 1/2, 1/2)$, then to produce y', technology t needs $a_0 \cdot (3, 4, 0)^T = 5$, while s needs $a_0 \cdot (0, 1, 3)^T = 2$. Thus, we should use technology s to produce y'.

Our analysis shows that some technologies are better than others, but what if we allow all activities to be used. Can we do any better? The answer is no.

For a given y, the following linear program gives the activity vector x that minimizes labor used.

$$\begin{aligned} \lambda(y) &= \min a_0 \cdot x \\ \text{s.t. } (B - A)x &\geq y \\ x &\geq 0 \end{aligned} \tag{$\mathcal{L}(y)$}$$

For a given y and a technology t, the following linear program gives the activity vector using technology t that minimizes labor used.

$$\begin{aligned} \lambda^t(y) &= \min a_0 \cdot x \\ \text{s.t. } (B - A)x &\geq y \\ x &\geq 0 \\ x_i &= 0 \text{ if } i \notin t \end{aligned} \tag{$\mathcal{L}^t(y)$}$$

First note that for a given y', if x is an optimal solution to $\mathcal{L}$, then there is a basic optimal solution which is an optimal solution to $\mathcal{L}^t$ for some t. That is, $\lambda(y') = \lambda^t(y')$ for some t. With the same reasoning, for a given y'', there is a technology s such that $\lambda(y'') = \lambda^s(y'')$. There is no reason for s and t to be the same in our reasoning, but the next theorem says that they are.

Theorem 2 (Non-substitution)**.** *If a GLM (A, B) satisfies the maintained assumptions, there is a technology t such that for all $y \geq 0$, $\lambda^t(y) = \lambda(y)$.*

Example 2. Let $y' = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$ and $y'' = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$. The problems are

$$\begin{aligned} \lambda(y') &= \min x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \text{s.t. } -x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 3 \\ x &\geq 0 \end{aligned} \tag{$\mathcal{L}(y')$}$$

$a_0\ x$.
$(B-A)x \geqslant y$.
$x \geqslant 0$.

$$\begin{aligned} \lambda(y'') &= \min x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \text{s.t. } -x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 4 \\ x &\geq 0 \end{aligned} \tag{$\mathcal{L}(y'')$}$$

The solutions are $x' = (0, 1, 3)^T$ and $x'' = (0, 1, 4)^T$, respectively. We see that technology $s = \{2, 3\}$ gives $\lambda^s(y') = \lambda(y')$ and also that $\lambda^s(y'') = \lambda(y'')$.

$\begin{pmatrix}0\\1\end{pmatrix}$ vs $\begin{pmatrix}1\\0\end{pmatrix}$

$$\min\ a_0 \cdot x \quad \text{s.t. } (B-A)x \geq y$$

Efficient production.

The non-substitution theorem says that technologies can be ranked by how efficient they are, where efficiency of a technology is measured by how much labor is used. Another interpretation of the non-substitution theorem is that when a GLM is productive, has one primary factor, and satisfies all our assumptions, restricting our usage to technology t with n activities does not alter the production possibility.

2 Problems

Problem 1 (Q7 of Linear Economic Models)**.**

Problem 2. Consider a GLM given by

(*)

$$B = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1/2 & 1/4 \end{bmatrix}$$

and suppose that there are two primary factors $a_P = (1,0)$ and $a_E = (0,1/2)$ with labor supply $L_P = 4$ and $L_E = 2$, respectively. Is there a technology that does not alter the production possibility? Compare this to the case with one primary factor $a_0 = (1,1/2)$ and $L = 6$.

$(A,B) \longrightarrow$ Productive ✓

↳ No joint prod

↳ All goods produced.

Tips:

1- PPS of all activities?

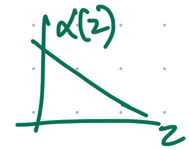
2- PPS of all technologies?

$\boxed{7}$ Countries : A , B

Goods : $z \in [0,1]$

Labor reqt : $\left(a(z)\ ,\ b(z)\right)_{z \in [0,1]}$ – only 1 input.

z is ordered so that $\alpha(z) = a(z)/b(z)$ strictly $\downarrow$ in z.



Output prices : $\left(p(z)\right)_{z \in [0,1]}$

Input prices : w_a , w_b

Input endowments : L_A , L_B

$\boxed{a}$ Convex support function of world PPS :

$$v(L_A, L_B) = \max_{x_a(\cdot),\, x_b(\cdot)} \int_0^1 p(z) \cdot \left[x_a(z) + x_b(z)\right] dz$$

$$\text{s.t.} \quad \int_0^1 a(z)\, x_a(z)\, dz \leq L_A \quad \text{----}(w_a)$$

$$\int_0^1 b(z)\, x_b(z)\, dz \leq L_B \quad \text{----}(w_b)$$

$$x_a(\cdot),\ x_b(\cdot) \geqslant 0$$

Complementary slackness :

$$w_a \left[\int_0^1 a(z)\, x_a(z)\, dz - L_A\right] = 0$$

$$w_b \left[\int_0^1 b(z)\, x_b(z)\, dz - L_B\right] = 0$$

$$\text{Dual} = \min_{\omega_a, \omega_b} \omega_a L_A + \omega_b L_B$$

$$\text{s.t.} \quad \left\{ \omega_a \cdot a(z) \geq p(z) \quad \dots \quad (x_a(z)) \right\} \; \forall z$$

$\Lambda_a(z) \leq 0$

$$\left\{ \omega_b \cdot b(z) \geq p(z) \quad \dots \quad (x_b(z)) \right\} \; \forall z$$

$$\omega_a, \omega_b \geq 0$$

Profit maximization condition: $\forall z \in [0,1]$,

$$x_a(z)\,[\underbrace{p(z) - \omega_a a(z)}_{=0}] = 0, \quad p(z) \leq \omega_a a(z)$$

$x_a(z) > 0$

$$x_b(z)\,[p(z) - \omega_b b(z)] = 0, \quad p(z) \leq \omega_b b(z)$$

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b If $x_a(\tilde{z}) > 0$, $x_b(\tilde{z}) > 0$ for some $\tilde{z} \in [0,1]$

then:

$$p(\tilde{z}) = \omega_a a(\tilde{z})$$

$$p(\tilde{z}) = \omega_b b(\tilde{z})$$

$$\Rightarrow \frac{a(\tilde{z})}{b(\tilde{z})} \equiv \alpha(\tilde{z}) = \frac{\omega_b}{\omega_a}$$

Statement: $\exists z^*$: A produced all $z < z^*$ & B all $z > z^*$.

B A.

c Altered proposition: $\exists z^* \in [0,1]$ such that

any $x_A^*(z) > 0$ has $z \geq z^*$ & any $x_B^*(z) > 0$ has $z \leq z^*$.

not all $z < z^*$ has $x_B^*(z) > 0$

<u>Pf:</u> Key ingredient:

i- Complementary slackness conditions

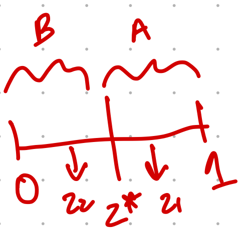
ii- $\alpha(\cdot)$ is strictly decreasing

Part (b) above says that for any good produced in both countries, $\alpha(\tilde{z}) = \dfrac{w_b}{w_a}$.

Since, $\alpha(\cdot)$ is strictly decreasing

$$\Rightarrow \exists \text{ atmost } \underline{\underline{\text{one}}} \quad \tilde{z} \in [0,1] : \alpha(\tilde{z}) = \frac{w_b}{w_a}.$$

Towards a contradiction of proposition, suppose:



$$\exists\ z_1, z_2 \text{ such that } \quad z_1 > z_2.$$

Let A produces z_2 and B produces z_1.

$$\div\ \frac{\cancel{w}_a\, a(z_1) \geq p(z_1)}{\cancel{w}_a\, a(z_2) = p(z_2)} \quad \& \quad \div\ \frac{\cancel{w}_b\, b(z_1) = p(z_1)}{\cancel{w}_b\, b(z_2) \geq p(z_2)}$$

$$\frac{a(z_1)}{a(z_2)} \geq \frac{p(z_1)}{p(z_2)} \quad \& \quad \frac{p(z_1)}{p(z_2)} \geq \frac{b(z_1)}{b(z)}.$$

$$\Rightarrow \quad \frac{a(z_1)}{b(z_1)} \geq \frac{a(z_2)}{b(z_2)} \quad \equiv \quad \alpha(z_1) \geq \alpha(z_2)$$

By strict decreasing: $\xRightarrow{z_1 > z_2} \alpha(z_1) < \alpha(z_2) \Rightarrow$ Contradiction

But suppose $x_a(z_1) > 0$, does this imply

$x_a(z_2) > 0 \quad \forall z_2 > z_1$?

$$x_a(z_1) > 0 \Rightarrow p(z_1) = w_a a(z_1).$$

It is possible that:

$$p(z_2) < w_a a(z_2) \Rightarrow x_a(z_2) = 0$$

$$\boxed{\frac{p(z_2)}{a(z_2)} < \frac{p(z_1)}{a(z_1)}}!$$

$\rightarrow x_a(z_2) = 0$. Not the case that all $z > z^*$ have $x_a(z) > 0$.

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d) See b. Notice that z^* is the only good that both A and B can produce simultaneously.

This happens when: $0 \leq z^* = \frac{w_b}{w_a} \leq 1$

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e) Identical consumers with Cobb-Douglas preferences

$\Rightarrow$ Use aggregate representation.

Consumer problem: $\max_{C_A} U_A\left[\left(C_A(z)\right)_{z\in[0,1]}\right]$.

$$\text{s.t.} \int_0^1 p(z) C_A(z)\, dz \leq \underbrace{w_A L_A}_{\text{selling labor in A.}}$$

$S(z) = \frac{p(z) C_i(z)}{w_i L_i}$

$U_A(\cdot)$ is Cobb Douglas

$$\implies \text{LNS} \implies \int_0^1 p(z)\, c_A(z)\, dz = w_A L_A .$$

$$\therefore \int_0^1 \underbrace{p(z)\, \frac{c_A(z)}{w_A L_A}}_{= s_A(z)} dz = 1 .$$

If $U_A = U_B$, given preferences are C-D:

$$\frac{p(z)\, c_A(z)}{w_A L_A} = \frac{p(z)\, c_B(z)}{w_B L_A}$$

$$s_A(z) = s_B(z) = s(z) .$$

$\boxed{f}$ Fraction spent on country A goods:

$$\tilde{\theta}(z^*) = \int_{z^*}^1 s(z)\, dz \qquad \left(\because \tilde{\theta} = 1 - \theta \text{ in p.s.}\right)$$

$$\theta(z) = \int_0^{z^*} s(z)\, dz .$$

Revenue in A $= \tilde{\theta}(z^*)\left[w_a L_a + w_b L_b\right]$

Labor expense in A $= w_a L_a$

Eqbm: $\tilde{\theta}(z^*)\, w_b L_b = w_a L_a \left[1 - \tilde{\theta}(z^*)\right]$

Revenue in B $= (1-\tilde{\theta}(z^*))[w_a L_a + w_b L_b]$

Labor expense in B $= w_b L_b$

Eqbm: $\tilde{\theta}(z^*) w_b L_b = (1-\tilde{\theta}(z^*)) w_a L_a$.

———— x ————

$\tilde{\theta} = \int_{z^*}^{1} \delta(z) dz$

$\boxed{g}$ $$\frac{w_b}{w_a} = \frac{L_a}{L_b} \frac{(1-\tilde{\theta}(z^*))}{\tilde{\theta}(z^*)} \equiv \beta(z^*)$$

$\boxed{h}$ $$\beta(z^*) = \frac{w_b}{w_a} = \alpha(z^*).$$

Diff. w.r.t. L_a $\left(\frac{d\alpha(z^*)}{dz^*} = \frac{d\beta(z^*)}{dz^*}\right)$

$$\alpha'(z^*) \cdot \frac{dz^*}{dL_a} = \frac{d}{dL_a}\left\{ \overset{u}{\left(\frac{L_a}{L_b}\right)} \overset{v}{\left(\frac{(1-\tilde{\theta}(z^*))}{\tilde{\theta}(z^*)}\right)} \right\}$$

$d\frac{u}{v}$. $\frac{v\,du - u\,dv}{v^2}$

$$= \overset{du}{\frac{1}{L_b}} \left[\overset{v}{\frac{1-\tilde{\theta}(z^*)}{\tilde{\theta}(z^*)}} \right] + \overset{u}{\frac{L_a}{L_b}} \left[\overset{dv}{\frac{\left(-\tilde{\theta}(z^*) - (1-\tilde{\theta}(z^*))\right) \frac{d\tilde{\theta}(z^*)}{dL_a}}{\tilde{\theta}(z^*)^2}} \right]$$

$\equiv \beta^*(z)$.

$$= \frac{1}{L_b}\left[\frac{1-\tilde{\theta}(z^*)}{\tilde{\theta}(z^*)}\right] - \frac{L_a}{L_b}\left(\frac{d\tilde{\theta}(z^*)/dz^* \cdot dz^*/dL_a}{\tilde{\theta}(z^*)^2}\right)$$

$$\tilde{\theta}(z^*) = \int_{z^*}^{1} s(z)\,dz = \int_{z^*}^{1} \frac{\overbrace{p(z)c(z)}^{\geq 0}}{w_A L_A + w_B L_B}\,dz.$$

Using Leibniz Rule:

$$\frac{d\tilde{\theta}(z^*)}{dz^*} = -1 \cdot s(z^*)$$

$$\therefore \alpha'(z^*)\frac{dz^*}{dL_a} = \frac{1}{L_b}\left[\frac{1-\tilde{\theta}(z^*)}{\tilde{\theta}(z^*)}\right] + \frac{L_a}{L_b}\left[\frac{s(z^*)}{\tilde{\theta}(z^*)^2}\right]\frac{dz^*}{dL_a}$$

$$\alpha'(z^*)L_b\tilde{\theta}(z^*)^2\frac{dz^*}{dL_a} = \tilde{\theta}(z^*)(1-\tilde{\theta}(z^*)) + L_a\, s(z^*)\frac{dz^*}{dL_a}$$

$$\boxed{\frac{dz^*}{dL_a} = \frac{\overbrace{\tilde{\theta}(z^*)}^{>0}\overbrace{\left[1-\tilde{\theta}(z^*)\right]}^{\geq 0}}{\underbrace{\alpha'(z^*)}_{<0}L_b\tilde{\theta}(z^*)^2 - L_a\underbrace{s(z^*)}_{\geq 0}}\ \ \geq 0.}$$

$$\frac{dz^*}{dL_a} < 0$$

Intuition: $L_a \uparrow \Rightarrow w_a \downarrow \Rightarrow \frac{w_b}{w_a}\uparrow$, $\alpha(z^*) = \frac{w_b}{w_a}$ & $\alpha'(\cdot) < 0$ $\Rightarrow z^* \downarrow$

———— x ————

Q2 Start by defining the PPS using all technologies

$A_{n\times m}$ $n=1$, $m=2$.

Each row of a_0 is a constraint.

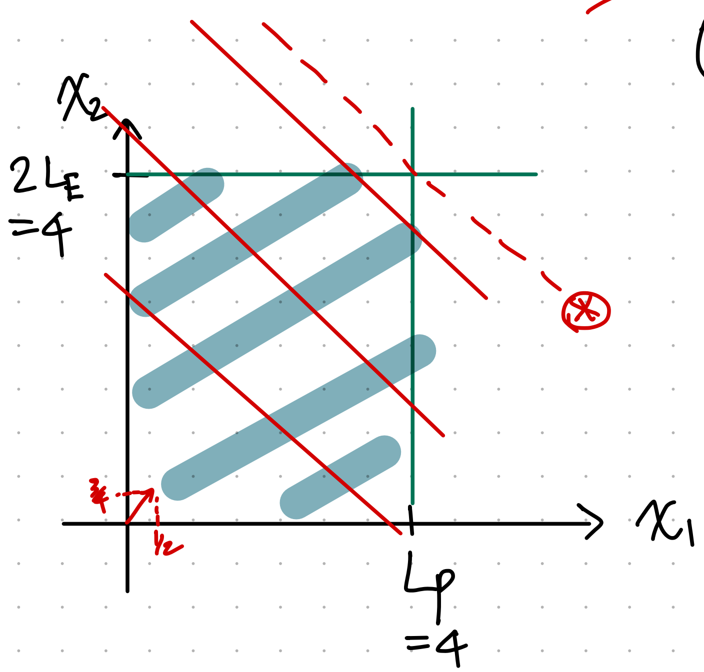
$$Y = \{\underset{1\times 1}{y} \geq 0 : (B-A)\underset{2\times 1}{\vec{x}} \geq y,\ \underset{2\times 2}{\underline{a_0}}\ \underset{2\times 1}{\vec{x}} \leq \underset{2\times 1}{\vec{L}},\ \vec{x} \geq 0\}$$

$B = [1, 1]$

$$\underline{a_0} = \begin{bmatrix} \vec{a}_P \\ \vec{a}_E \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix};\quad \vec{L} = \begin{bmatrix} L_P \\ L_E \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$Y = \left\{y \geq 0 : y \leq \left(\frac{x_1}{2} + \frac{3x_2}{4}\right),\ 0 \leq x_1 \leq 4,\ 0 \leq \frac{x_2}{2} \leq 2\right\}$$

$\underline{a_0} : 1\times m$



Set $x_1 = x_2 = 4$.

$$Y = \frac{4}{2} + \frac{3\times 4}{4} = $$

$$= 2 + 3 = \underline{\underline{5}}$$

Suppose we only use technology 1,

$x_1 = 4$, $x_2 = 0$ $\Rightarrow$ $y = \underline{\underline{2}}$.

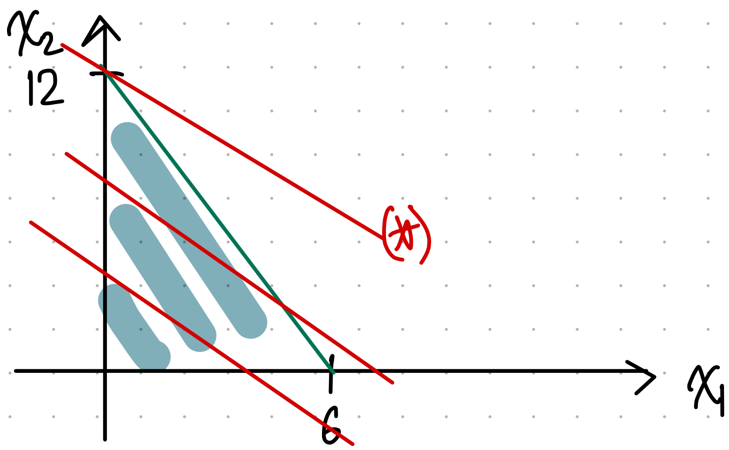
Suppose we only use technology 2,

$x_1 = 0$, $x_2 = 4$ $\Rightarrow$ $y = \underline{\underline{3}}$.

Thus, $\nexists\ \tau \in \{\{1\},\{2\}\}$ such that the PPS is unchanged

If $\vec{a}_0 = (1, 1/2)$; $L = 6$

$$Y = \left\{ y \geq 0 : \ y \leq \frac{x_1}{2} + \frac{3x_2}{4}, \ x_1 + \frac{x_2}{2} \leq 6, \ \vec{x} \geq 0 \right\}$$



$x_1^* = 0$; $\underline{x_2^* = 12}$.

$\underline{y = 9}$

$\tau = \{2\}$ produces same PPS.